

Chapter 2

Vector Functions of a Real Variable

LINES AND PLANES

Let $\mathbf{a}$ and $\mathbf{u}$ be vectors in E^3 with $\mathbf{u} \neq \mathbf{0}$. By the *straight line through $\mathbf{a}$ parallel to $\mathbf{u}$* we mean the set of $\mathbf{x}$ in E^3 which can be represented by

$$\mathbf{x} = k\mathbf{u} + \mathbf{a}, \quad -\infty < k < \infty \quad (2.1)$$

or in component form,

$$x_1 = ku_1 + a_1, \quad x_2 = ku_2 + a_2, \quad x_3 = ku_3 + a_3, \quad -\infty < k < \infty \quad (2.2)$$

The equations (2.1) or (2.2) are called the parametric equations of the line. We say that the point $\mathbf{x}$ *generates* the line as the *parameter* k varies over the real line. Any vector which is linearly dependent on $\mathbf{u}$ will be said to be *parallel* to this line. Two lines will be said to be parallel if their respective vectors $\mathbf{u}$ are linearly dependent.

Example 2.1.

The parametric equation of the line through $\mathbf{a} = \mathbf{e}_1 + 2\mathbf{e}_2$ parallel to $\mathbf{u} = \mathbf{e}_1 - \mathbf{e}_3$ is

$$\mathbf{x} = k\mathbf{u} + \mathbf{a} = k(\mathbf{e}_1 - \mathbf{e}_3) + (\mathbf{e}_1 + 2\mathbf{e}_2) = (k+1)\mathbf{e}_1 + 2\mathbf{e}_2 - k\mathbf{e}_3$$

or $x_1 = k+1$, $x_2 = 2$, $x_3 = -k$.

Example 2.2.

If $\mathbf{a}$ and $\mathbf{b}$ are distinct points on a line, then $\mathbf{b} - \mathbf{a}$ is a nonzero vector parallel to the line. Thus the equation of the line through $\mathbf{a}$ and $\mathbf{b}$ is $\mathbf{x} = k(\mathbf{b} - \mathbf{a}) + \mathbf{a}$, or

$$x_1 = k(b_1 - a_1) + a_1, \quad x_2 = k(b_2 - a_2) + a_2, \quad x_3 = k(b_3 - a_3) + a_3$$

By the *plane through $\mathbf{a}$ parallel to two independent vectors $\mathbf{u}$ and $\mathbf{v}$* we mean the set of $\mathbf{x}$ in E^3 which can be represented by

$$\mathbf{x} = h\mathbf{u} + k\mathbf{v} + \mathbf{a}, \quad -\infty < h < \infty, \quad -\infty < k < \infty \quad (2.3)$$

or, equating components,

$$x_1 = hu_1 + kv_1 + a_1, \quad x_2 = hu_2 + kv_2 + a_2, \quad x_3 = hu_3 + kv_3 + a_3 \quad (2.4)$$

The equations (2.3) and (2.4) are called the parametric equations of the plane. We say that $\mathbf{x}$ generates the plane as the parameters h and k vary independently over the real numbers. A vector will be said to be parallel to the plane if it is linearly dependent upon $\mathbf{u}$ and $\mathbf{v}$, and it will be said to be normal to the plane if it is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

If $\mathbf{n}$ is a nonzero vector normal to the plane $\mathbf{x} = h\mathbf{u} + k\mathbf{v} + \mathbf{a}$, then the point $\mathbf{x}$ lies on the plane if and only if $\mathbf{x} - \mathbf{a}$ is orthogonal to $\mathbf{n}$, or

$$(\mathbf{x} - \mathbf{a}) \cdot \mathbf{n} = 0 \quad (2.5)$$

In terms of the components of $\mathbf{x}$, $\mathbf{a}$ and $\mathbf{n}$, this becomes

$$(x_1 - a_1)n_1 + (x_2 - a_2)n_2 + (x_3 - a_3)n_3 = 0 \quad (2.6)$$

If a, b, c are noncollinear points on a plane, then $b - a$ and $c - a$ are linearly independent vectors parallel to the plane and $(b - a) \times (c - a)$ is a nonzero vector normal to the plane as indicated in Fig. 2-1. It follows from equation (2.5) that the equation of the plane through a, b, c is

$$[(x - a)(b - a)(c - a)] = 0 \quad (2.7)$$

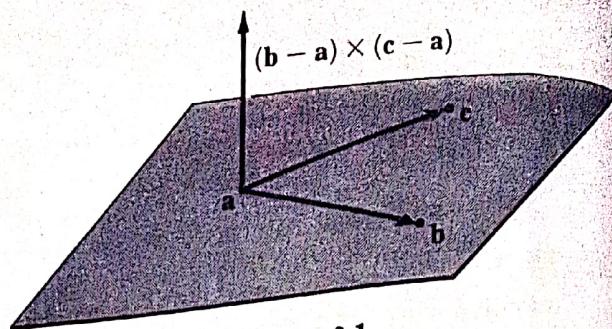


Fig. 2-1

Example 2.3.

The parametric equation of the plane through $a = e_2$ parallel to $u = e_1$ and $v = -e_1 + e_3$ is

$$x = hu + kv + a = (h - k)e_1 + e_2 + ke_3 \quad \text{or} \quad x_1 = h - k, \quad x_2 = 1, \quad x_3 = k$$

The plane is also given by

$$[(x - a)uv] = \det \begin{pmatrix} x_1 & 1 & -1 \\ x_2 - 1 & 0 & 0 \\ x_3 & 0 & 1 \end{pmatrix} = 0 \quad \text{or} \quad x_2 - 1 = 0$$

NEIGHBORHOODS

Local properties of functions are conveniently described in terms of the concept of a spherical neighborhood. Namely, the ϵ -open sphere or ϵ -spherical neighborhood of a vector a , denoted by $S_\epsilon(a)$, is the set of x satisfying $|x - a| < \epsilon$. As shown in Fig. 2-2, a point x is in $S_\epsilon(a)$ if and only if x is in the interior of the sphere of radius ϵ about a . In E^2 , $S_\epsilon(a)$ is the interior of the circle of radius ϵ about a ; and in E^1 , $S_\epsilon(a)$ is the open interval of length 2ϵ with a at its center, as shown in Fig. 2-3.

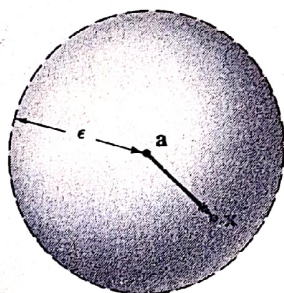


Fig. 2-2

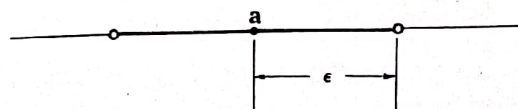


Fig. 2-3

It is also convenient to consider a spherical neighborhood of a less a itself. The set $S_\epsilon(a)$ excluding a is called the ϵ -deleted spherical neighborhood of a and is denoted by $S'_\epsilon(a)$. Since $|x - a| = 0$ if and only if $x = a$, $S'_\epsilon(a)$ consists of the vectors x satisfying $0 < |x - a| < \epsilon$.

Example 2.4.

The $1/10$ spherical neighborhood of the vector $a = e_1 + 2e_2 + 3e_3$, i.e. $S_{1/10}(e_1 + 2e_2 + 3e_3)$, consists of the vectors $x = x_1e_1 + x_2e_2 + x_3e_3$ satisfying

$$|x - a| = [(x_1 - 1)^2 + (x_2 - 2)^2 + (x_3 - 3)^2]^{1/2} < 1/10 \quad \text{or} \quad (x_1 - 1)^2 + (x_2 - 2)^2 + (x_3 - 3)^2 < 1/100$$

Example 2.5.

$S_{1/100}(5)$ on E^1 is the set of numbers x satisfying $|x - 5| < 1/100$ or $5 - 1/100 < x < 5 + 1/100$. Note that $S_{1/100}$ is the open interval of length $1/50$ centered about 5.

VECTOR FUNCTIONS

The assignment of a vector $\mathbf{f}(t)$ to each real number t of a set of real numbers S defines a *vector function* $\mathbf{f}$ of the single real variable t in S . As in the case of scalar functions of a real variable, the set S is called the *domain of definition* of $\mathbf{f}$; and the set of assigned vectors, denoted by $\mathbf{f}(S)$, is called the *image* of $\mathbf{f}$.

Example 2.6.

Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be fixed vectors in space. The equation

$$\mathbf{f}(t) = \mathbf{a} - 2t\mathbf{b} + t^2\mathbf{c}, \quad -2 \leq t \leq 2$$

defines a vector function of t with domain $-2 \leq t \leq 2$. A table of some assigned vectors is

t	-2	-1	0	1	2
$\mathbf{f}(t)$	$\mathbf{a} + 4\mathbf{b} + 4\mathbf{c}$	$\mathbf{a} + 2\mathbf{b} + \mathbf{c}$	$\mathbf{a}$	$\mathbf{a} - 2\mathbf{b} + \mathbf{c}$	$\mathbf{a} - 4\mathbf{b} + 4\mathbf{c}$

Example 2.7.

In Example 2.6 suppose that $\mathbf{a} = \mathbf{e}_1 + 2\mathbf{e}_2$, $\mathbf{b} = \mathbf{e}_2 - \mathbf{e}_3$, $\mathbf{c} = \mathbf{e}_1 - \mathbf{e}_3$. Then

$$\mathbf{f}(t) = (\mathbf{e}_1 + 2\mathbf{e}_2) - 2t(\mathbf{e}_2 - \mathbf{e}_3) + t^2(\mathbf{e}_1 - \mathbf{e}_3) = (1 + t^2)\mathbf{e}_1 + (2 - 2t)\mathbf{e}_2 + (2t - t^2)\mathbf{e}_3$$

Here $\mathbf{f}$ is expressed in terms of the three scalar functions $f_1(t) = 1 + t^2$, $f_2(t) = 2 - 2t$, $f_3(t) = 2t - t^2$, its components with respect to $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$.

As indicated in the example above, $\mathbf{f}(t)$ uniquely determines three scalar functions $f_1(t), f_2(t), f_3(t)$, its components with respect to the basis. Conversely, three scalar functions $f_1(t), f_2(t), f_3(t)$ on a common domain S uniquely define a vector function

$$\mathbf{f}(t) = f_1(t)\mathbf{e}_1 + f_2(t)\mathbf{e}_2 + f_3(t)\mathbf{e}_3$$

on S whose components with respect to $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ are f_1, f_2, f_3 .

Vector functions will be used to define curves. Let $\mathbf{x} = \mathbf{f}(t)$; then as t varies, the point $\mathbf{x}$ will trace out a curve, as shown in Fig. 2-4. The equation $\mathbf{x} = \mathbf{f}(t)$ or, componentwise, the three scalar equations

$$x_1 = f_1(t), \quad x_2 = f_2(t), \quad x_3 = f_3(t)$$

will be called a *parametric representation* of the curve, and the variable t will be the *parameter*.

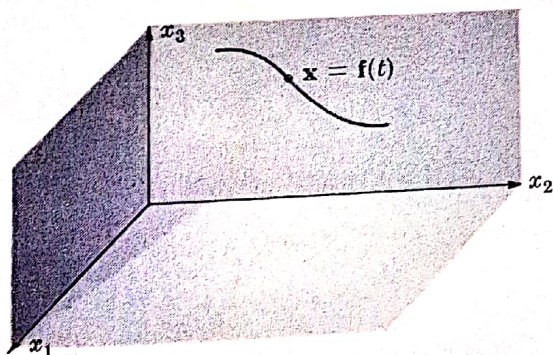


Fig. 2-4

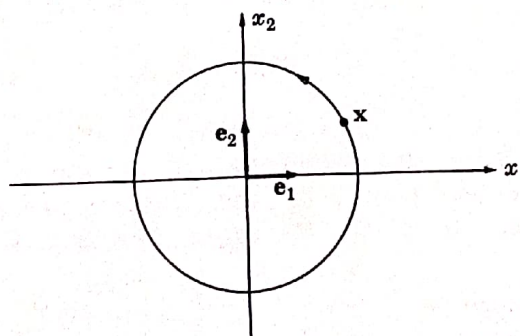


Fig. 2-5

Example 2.8.

The equation $\mathbf{x} = a(\cos t)\mathbf{e}_1 + a(\sin t)\mathbf{e}_2$ or $x_1 = a \cos t$, $x_2 = a \sin t$, $a > 0$, $0 \leq t \leq 2\pi$ is a parametric representation of the circle of radius a about the origin. As t increases through the interval $0 \leq t \leq 2\pi$, the circle is traced in a counterclockwise direction, as shown in Fig. 2-5 above.

Note that the existence of a limit at t_0 is a *local* property of a function, depending only on the nature of the function in a *deleted neighborhood* of t_0 . Moreover, $f(t)$ need not be defined at t_0 . For example, its domain could be the open interval $a < t < t_0$.

Example 2.10.

Let $f(t) = a = \text{constant}$. Then for any t_0 , $\lim_{t \rightarrow t_0} f(t) = a$. For $f(t) \equiv a$ is in every $S_\epsilon(a)$ for all t and hence for all $S_\delta(t_0)$ for all t_0 .

Example 2.11.

The function $\mathbf{x} = f(t) = \begin{cases} t\mathbf{e}_1 + \mathbf{e}_2, & t \geq 0 \\ t\mathbf{e}_1 - \mathbf{e}_2, & t < 0 \end{cases}$

or $x_1 = f_1(t) = t$, $x_2 = f_2(t) = \begin{cases} 1, & t \geq 0 \\ -1, & t < 0 \end{cases}$

shown in Fig. 2-9, does not have a limit as $t \rightarrow 0$; for any point $\mathbf{L}$ has a neighborhood $S_\epsilon(\mathbf{L})$ which does not intersect both the line $x_2 = 1$ and the line $x_2 = -1$. (For example, as shown, $S_{1/2}(0, 1)$ will not include points on $x_2 = -1$.) For these $S_\epsilon(\mathbf{L})$ there will not exist a $\delta > 0$ such that for all $0 < |t| < \delta$ the points $\mathbf{x} = f(t)$ are in $S_\epsilon(\mathbf{L})$. Since $\mathbf{L}$ is arbitrary, there is no limit. On the other hand, the function does have a limit for any other choice of t_0 . For example, as $t \rightarrow \frac{1}{2}$ the limit is $\frac{1}{2}\mathbf{e}_1 + \mathbf{e}_2$.

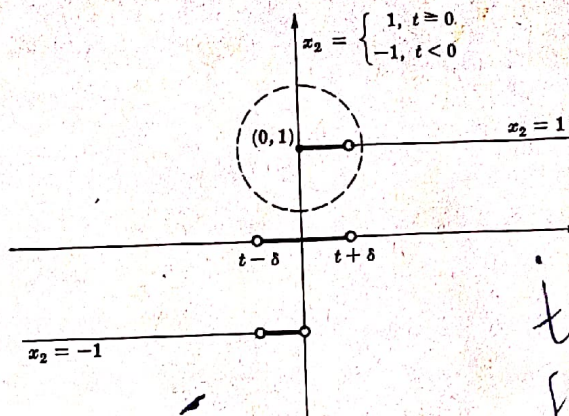


Fig. 2-9

Now, we recall that a scalar function $g(t) \rightarrow 0$ as $t \rightarrow t_0$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $|g(t)| < \epsilon$ for t in $S_\delta(t_0)$. If we let $g(t) = |\mathbf{f}(t) - \mathbf{L}|$, then $|g(t)| = |\mathbf{f}(t) - \mathbf{L}| < \epsilon$ if and only if $\mathbf{f}(t)$ is in $S_\epsilon(\mathbf{L})$. Thus we have the important

Theorem 2.1. $\mathbf{f}(t) \rightarrow \mathbf{L}$ as $t \rightarrow t_0$ iff $|\mathbf{f}(t) - \mathbf{L}| \rightarrow 0$ as $t \rightarrow t_0$.

Example 2.12.

$\lim_{t \rightarrow 1} (t^2\mathbf{e}_1 - (t+1)\mathbf{e}_2) = \mathbf{e}_1 - 2\mathbf{e}_2$, since

$$\lim_{t \rightarrow 1} |\mathbf{f}(t) - \mathbf{L}| = \lim_{t \rightarrow 1} |(t^2 - 1)\mathbf{e}_1 - (t - 1)\mathbf{e}_2| = \lim_{t \rightarrow 1} [(t^2 - 1)^2 + (t - 1)^2]^{1/2} = 0$$

Finally, suppose $\mathbf{f}(t) \rightarrow \mathbf{L}$ as $t \rightarrow t_0$. Then for an arbitrary $\epsilon > 0$, there exists $\delta > 0$ such that $|\mathbf{f}(t) - \mathbf{L}| < \epsilon$ for t in $S_\delta(t_0)$. Hence for t in $S_\delta(t_0)$,

$$|\mathbf{f}(t)| = |\mathbf{f}(t) - \mathbf{L} + \mathbf{L}| \leq |\mathbf{f}(t) - \mathbf{L}| + |\mathbf{L}| \leq M$$

where $M = \text{Max}(\epsilon, |\mathbf{f}(t_0) - \mathbf{L}|) + |\mathbf{L}|$. Thus we have

Theorem 2.2. If $\mathbf{f}(t)$ has a limit as $t \rightarrow t_0$, then $\mathbf{f}(t)$ is bounded at t_0 .

PROPERTIES OF LIMITS

Suppose $\lim_{t \rightarrow t_0} f_i(t) = L_i$, $i = 1, 2, 3$; then

$$\lim_{t \rightarrow t_0} [f_1(t)\mathbf{e}_1 + f_2(t)\mathbf{e}_2 + f_3(t)\mathbf{e}_3] = L_1\mathbf{e}_1 + L_2\mathbf{e}_2 + L_3\mathbf{e}_3$$

For, let $\mathbf{f}(t) = f_1(t)\mathbf{e}_1 + f_2(t)\mathbf{e}_2 + f_3(t)\mathbf{e}_3$ and $\mathbf{L} = L_1\mathbf{e}_1 + L_2\mathbf{e}_2 + L_3\mathbf{e}_3$; then

$$\begin{aligned} \lim_{t \rightarrow t_0} |\mathbf{f}(t) - \mathbf{L}| &= \lim_{t \rightarrow t_0} |(f_1(t)\mathbf{e}_1 + f_2(t)\mathbf{e}_2 + f_3(t)\mathbf{e}_3) - (L_1\mathbf{e}_1 + L_2\mathbf{e}_2 + L_3\mathbf{e}_3)| \\ &= \lim_{t \rightarrow t_0} [(f_1(t) - L_1)^2 + (f_2(t) - L_2)^2 + (f_3(t) - L_3)^2]^{1/2} \\ &= 0 \end{aligned}$$

The converse of the above is also true. Namely, we have

Theorem 2.3. The function $\mathbf{f}(t) = f_1(t)\mathbf{e}_1 + f_2(t)\mathbf{e}_2 + f_3(t)\mathbf{e}_3$ has a limit as $t \rightarrow t_0$ if and only if $f_i(t)$, $i = 1, 2, 3$, have limits as $t \rightarrow t_0$, in which case

$$\lim_{t \rightarrow t_0} \mathbf{f}(t) = \left(\lim_{t \rightarrow t_0} f_1(t) \right) \mathbf{e}_1 + \left(\lim_{t \rightarrow t_0} f_2(t) \right) \mathbf{e}_2 + \left(\lim_{t \rightarrow t_0} f_3(t) \right) \mathbf{e}_3$$

Example 2.13.

$$\lim_{t \rightarrow 0} ((\sin t)\mathbf{e}_1 + (\cos t)\mathbf{e}_2 + t\mathbf{e}_3) = \left(\lim_{t \rightarrow 0} \sin t \right) \mathbf{e}_1 + \left(\lim_{t \rightarrow 0} \cos t \right) \mathbf{e}_2 + \left(\lim_{t \rightarrow 0} t \right) \mathbf{e}_3 = \mathbf{e}_2$$

Example 2.14.

Let $\mathbf{f}(t) = t^2\mathbf{e}_1 + t\mathbf{e}_2$. Then

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{\mathbf{f}(2+h) - \mathbf{f}(2)}{h} &= \lim_{h \rightarrow 0} \frac{((2+h)^2\mathbf{e}_1 + (2+h)\mathbf{e}_2) - (4\mathbf{e}_1 + 2\mathbf{e}_2)}{h} \\ &= \lim_{h \rightarrow 0} \left[\frac{((2+h)^2 - 4)\mathbf{e}_1}{h} + \frac{h\mathbf{e}_2}{h} \right] = 4\mathbf{e}_1 + \mathbf{e}_2 \end{aligned}$$

Now suppose $\mathbf{f}(t) \rightarrow \mathbf{L}$ as $t \rightarrow t_0$; then $|\mathbf{f}(t)| \rightarrow |\mathbf{L}|$ as $t \rightarrow t_0$. For, letting $\mathbf{f}(t) = f_1(t)\mathbf{e}_1 + f_2(t)\mathbf{e}_2 + f_3(t)\mathbf{e}_3$ and $\mathbf{L} = L_1\mathbf{e}_1 + L_2\mathbf{e}_2 + L_3\mathbf{e}_3$,

$$\begin{aligned} \lim_{t \rightarrow t_0} |\mathbf{f}(t)| &= \lim_{t \rightarrow t_0} [f_1^2(t) + f_2^2(t) + f_3^2(t)]^{1/2} \\ &= \left[\left(\lim_{t \rightarrow t_0} f_1(t) \right)^2 + \left(\lim_{t \rightarrow t_0} f_2(t) \right)^2 + \left(\lim_{t \rightarrow t_0} f_3(t) \right)^2 \right]^{1/2} \\ &= [L_1^2 + L_2^2 + L_3^2]^{1/2} = |\mathbf{L}| \end{aligned}$$

Note, however, that the converse of the above is not true. That is, $|\mathbf{f}(t)|$ may have a limit even though $\mathbf{f}(t)$ does not. This happens in Example 2.11 at $t_0 = 0$.

We state the above result formally as

Theorem 2.4. If $\mathbf{f}(t) \rightarrow \mathbf{L}$ as $t \rightarrow t_0$, then $|\mathbf{f}(t)| \rightarrow |\mathbf{L}|$ as $t \rightarrow t_0$.

Finally, we have: If $\lim_{t \rightarrow t_0} \mathbf{f}(t) = \mathbf{L}$, $\lim_{t \rightarrow t_0} \mathbf{g}(t) = \mathbf{M}$ and $\lim_{t \rightarrow t_0} h(t) = N$, then

$$[\mathbf{H}_1] \quad \lim_{t \rightarrow t_0} (\mathbf{f}(t) + \mathbf{g}(t)) = \lim_{t \rightarrow t_0} \mathbf{f}(t) + \lim_{t \rightarrow t_0} \mathbf{g}(t) = \mathbf{L} + \mathbf{M}$$

$$[\mathbf{H}_2] \quad \lim_{t \rightarrow t_0} (h(t)\mathbf{g}(t)) = \lim_{t \rightarrow t_0} h(t) \lim_{t \rightarrow t_0} \mathbf{g}(t) = N\mathbf{M}$$

$$[\mathbf{H}_3] \quad \text{If } N \neq 0, \text{ then } \lim_{t \rightarrow t_0} (\mathbf{f}(t)/h(t)) = \lim_{t \rightarrow t_0} \mathbf{f}(t) / \lim_{t \rightarrow t_0} h(t) = \mathbf{L}/N.$$

$$[\mathbf{H}_4] \quad \lim_{t \rightarrow t_0} (\mathbf{f}(t) \cdot \mathbf{g}(t)) = \lim_{t \rightarrow t_0} \mathbf{f}(t) \cdot \lim_{t \rightarrow t_0} \mathbf{g}(t) = \mathbf{L} \cdot \mathbf{M}$$

$$[\mathbf{H}_5] \quad \lim_{t \rightarrow t_0} (\mathbf{f}(t) \times \mathbf{g}(t)) = \lim_{t \rightarrow t_0} \mathbf{f}(t) \times \lim_{t \rightarrow t_0} \mathbf{g}(t) = \mathbf{L} \times \mathbf{M}.$$

$$[\mathbf{H}_6] \quad \text{If } \lim_{t \rightarrow t_0} \mathbf{f}(t) = \mathbf{f}(t_0) \text{ and } \lim_{\theta \rightarrow \theta_0} h(\theta) = t_0, \text{ then } \lim_{\theta \rightarrow \theta_0} \mathbf{f}(h(\theta)) = \mathbf{f} \left(\lim_{\theta \rightarrow \theta_0} h(\theta) \right) = \mathbf{f}(t_0).$$

Example 2.15.

Let $\lim_{t \rightarrow t_0} \mathbf{f}(t) = \mathbf{L}$, $\lim_{t \rightarrow t_0} \mathbf{g}(t) = \mathbf{M}$, $\lim_{t \rightarrow t_0} h(t) = N$. Then

$$\begin{aligned} \lim_{t \rightarrow t_0} [\mathbf{f}(t) \mathbf{g}(t) h(t)] &= \lim_{t \rightarrow t_0} (\mathbf{f}(t) \cdot \mathbf{g}(t) \times h(t)) = \lim_{t \rightarrow t_0} \mathbf{f}(t) \cdot \lim_{t \rightarrow t_0} (\mathbf{g}(t) \times h(t)) \\ &= \lim_{t \rightarrow t_0} \mathbf{f}(t) \cdot \lim_{t \rightarrow t_0} \mathbf{g}(t) \times \lim_{t \rightarrow t_0} h(t) = [\mathbf{LMN}] \end{aligned}$$

CONTINUITY

A vector function $\mathbf{f}(t)$ defined at t_0 is *continuous* at t_0 if for every $\epsilon > 0$ there exists a $\delta > 0$, depending on ϵ , such that $\mathbf{f}(t)$ is in $S_\epsilon(\mathbf{f}(t_0))$ for all t in $S_\delta(t_0)$; or, equivalently, $\mathbf{f}(t)$ is continuous at t_0 if

$$\lim_{t \rightarrow t_0} \mathbf{f}(t) = \mathbf{f}(t_0) \quad (2.8)$$

The function $\mathbf{f}(t)$ is said to be continuous on I if it is continuous at all $t = t_0$ in I .

It follows from Theorem 2.3 that $\mathbf{f}(t)$ is continuous if and only if its components $f_i(t)$, $i = 1, 2, 3$, are continuous. Also it follows from $[\mathbf{H}_1]$ through $[\mathbf{H}_6]$ that the sum, product, and scalar and vector products of continuous functions are continuous and that a continuous function of a continuous function is continuous.

Finally, we note that (2.8) is equivalent to

$$\lim_{t \rightarrow t_0} (\mathbf{f}(t) - \mathbf{f}(t_0)) = \mathbf{0}$$

or, if we let $h = t - t_0$,

$$\lim_{h \rightarrow 0} (\mathbf{f}(t_0 + h) - \mathbf{f}(t_0)) = \mathbf{0}$$

Example 2.16.

Let $\mathbf{f}(t) = \mathbf{a} + \mathbf{b}t + \mathbf{c}t^2$ with $\mathbf{a}, \mathbf{b}, \mathbf{c} = \text{constants}$. Then

$$\lim_{t \rightarrow t_0} \mathbf{f}(t) = \lim_{t \rightarrow t_0} (\mathbf{a} + \mathbf{b}t + \mathbf{c}t^2) = \mathbf{a} + \mathbf{b}t_0 + \mathbf{c}t_0^2 = \mathbf{f}(t_0)$$

Hence $\mathbf{f}(t)$ is continuous for all t .

Example 2.17.

Let $\mathbf{f}(t) = \begin{cases} \frac{t^2 - 1}{t - 1} \mathbf{e}_1 + t^3 \mathbf{e}_2, & t \neq 1 \\ 2\mathbf{e}_1 + \mathbf{e}_2, & t = 1 \end{cases}$. Then $\mathbf{f}(t)$ is continuous for all t . For $t_0 \neq 1$,

$$\lim_{t \rightarrow t_0} \mathbf{f}(t) = \lim_{t \rightarrow t_0} \left(\frac{t^2 - 1}{t - 1} \mathbf{e}_1 + t^3 \mathbf{e}_2 \right) = \frac{t_0^2 - 1}{t_0 - 1} \mathbf{e}_1 + t_0^3 \mathbf{e}_2 = \mathbf{f}(t_0)$$

For $t_0 = 1$,

$$\lim_{t \rightarrow 1} \mathbf{f}(t) = \lim_{t \rightarrow 1} \left(\frac{t^2 - 1}{t - 1} \mathbf{e}_1 + t^3 \mathbf{e}_2 \right) = \lim_{t \rightarrow 1} ((t+1)\mathbf{e}_1 + t^3 \mathbf{e}_2) = 2\mathbf{e}_1 + \mathbf{e}_2 = \mathbf{f}(1)$$

Example 2.18.

The function $\mathbf{f}(t) = \begin{cases} t\mathbf{e}_1 + \mathbf{e}_2, & t \geq 0 \\ t\mathbf{e}_1 - \mathbf{e}_2, & t < 0 \end{cases}$ in Example 2.11 is continuous at all t except $t = 0$ where the limit does not exist.

DIFFERENTIATION

The limit

$$\mathbf{f}'(t_0) = \lim_{t \rightarrow t_0} \frac{\mathbf{f}(t) - \mathbf{f}(t_0)}{t - t_0} \quad (2.9)$$

if it exists, defines the *derivative* of $\mathbf{f}(t)$ at $t = t_0$. If $\mathbf{f}'(t_0)$ exists, we say $\mathbf{f}(t)$ is *differentiable* at t_0 .

Observe that if we substitute $t = t_0 + \Delta t$ in the above, the derivative at t_0 is also given as

$$\mathbf{f}'(t_0) = \lim_{\Delta t \rightarrow 0} \frac{\mathbf{f}(t_0 + \Delta t) - \mathbf{f}(t_0)}{\Delta t} \quad (2.10)$$

Example 2.19.

Let $\mathbf{f}(t) = \mathbf{a} + \mathbf{b}t + \mathbf{c}t^2$, with $\mathbf{a}, \mathbf{b}, \mathbf{c} = \text{constants}$. Then

$$\begin{aligned} \mathbf{f}'(t_0) &= \lim_{\Delta t \rightarrow 0} \frac{\mathbf{f}(t_0 + \Delta t) - \mathbf{f}(t_0)}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{[\mathbf{a} + \mathbf{b}(t_0 + \Delta t) + \mathbf{c}(t_0 + \Delta t)^2] - (\mathbf{a} + \mathbf{b}t_0 + \mathbf{c}t_0^2)}{\Delta t} \\ &= \lim_{\Delta t \rightarrow 0} \frac{\mathbf{b}\Delta t + 2\mathbf{c}t_0\Delta t + \mathbf{c}(\Delta t)^2}{\Delta t} = \lim_{\Delta t \rightarrow 0} (\mathbf{b} + 2\mathbf{c}t_0 + \mathbf{c}\Delta t) = \mathbf{b} + 2\mathbf{c}t_0 \end{aligned}$$

Thus $\mathbf{f}(t)$ is differentiable at t_0 , with derivative $\mathbf{f}'(t_0) = \mathbf{b} + 2\mathbf{c}t_0$.

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Now, if $\mathbf{f}(t) = f_1(t)\mathbf{e}_1 + f_2(t)\mathbf{e}_2 + f_3(t)\mathbf{e}_3$, then it follows from Theorem 2.3 that

$$\begin{aligned} \mathbf{f}'(t_0) &= \lim_{t \rightarrow t_0} \frac{\mathbf{f}(t) - \mathbf{f}(t_0)}{t - t_0} \\ &= \lim_{t \rightarrow t_0} \left[\frac{f_1(t) - f_1(t_0)}{t - t_0} \mathbf{e}_1 + \frac{f_2(t) - f_2(t_0)}{t - t_0} \mathbf{e}_2 + \frac{f_3(t) - f_3(t_0)}{t - t_0} \mathbf{e}_3 \right] \\ &= \left[\lim_{t \rightarrow t_0} \frac{f_1(t) - f_1(t_0)}{t - t_0} \right] \mathbf{e}_1 + \left[\lim_{t \rightarrow t_0} \frac{f_2(t) - f_2(t_0)}{t - t_0} \right] \mathbf{e}_2 + \left[\lim_{t \rightarrow t_0} \frac{f_3(t) - f_3(t_0)}{t - t_0} \right] \mathbf{e}_3 \\ &= f_1'(t_0)\mathbf{e}_1 + f_2'(t_0)\mathbf{e}_2 + f_3'(t_0)\mathbf{e}_3 \end{aligned}$$

Thus we have

Theorem 2.5. A function $\mathbf{f}(t) = f_1(t)\mathbf{e}_1 + f_2(t)\mathbf{e}_2 + f_3(t)\mathbf{e}_3$ is differentiable at t_0 if and only if each component $f_i(t)$, $i = 1, 2, 3$, is differentiable at t_0 , in which case

$$\mathbf{f}'(t_0) = f_1'(t_0)\mathbf{e}_1 + f_2'(t_0)\mathbf{e}_2 + f_3'(t_0)\mathbf{e}_3$$

If $\mathbf{f}(t)$ is differentiable on an interval I , then $\mathbf{f}'(t)$ is again a vector function on I which may again be differentiable. This will give the second order derivative of $\mathbf{f}(t)$, denoted by $\mathbf{f}''(t)$. Higher order derivatives are defined similarly.

As with scalar functions, if $\mathbf{u} = \mathbf{f}(t)$, we use the notation

$$\mathbf{u}' = \frac{d\mathbf{u}}{dt} = \mathbf{f}'(t), \quad \mathbf{u}'' = \frac{d}{dt} \left(\frac{d\mathbf{u}}{dt} \right) = \frac{d^2\mathbf{u}}{dt^2} = \mathbf{f}''(t), \quad \text{etc.}$$

Example 2.20.

If $\mathbf{u} = (t^3 + 2t)\mathbf{e}_1 + (\sin t)\mathbf{e}_2 + e^t\mathbf{e}_3$, then

$$\mathbf{u}' = \frac{d\mathbf{u}}{dt} = \frac{d}{dt}(t^3 + 2t)\mathbf{e}_1 + \frac{d}{dt}(\sin t)\mathbf{e}_2 + \frac{d}{dt}(e^t)\mathbf{e}_3 = (3t^2 + 2)\mathbf{e}_1 + (\cos t)\mathbf{e}_2 + e^t\mathbf{e}_3$$

$$\mathbf{u}'' = \frac{d}{dt} \left(\frac{d\mathbf{u}}{dt} \right) = \frac{d}{dt}(3t^2 + 2)\mathbf{e}_1 + \frac{d}{dt}(\cos t)\mathbf{e}_2 + \frac{d}{dt}(e^t)\mathbf{e}_3 = 6t\mathbf{e}_1 - (\sin t)\mathbf{e}_2 + e^t\mathbf{e}_3$$

$$\mathbf{u}''' = \frac{d}{dt} \frac{d^2\mathbf{u}}{dt^2} = \frac{d}{dt}(6t)\mathbf{e}_1 - \frac{d}{dt}(\sin t)\mathbf{e}_2 + \frac{d}{dt}(e^t)\mathbf{e}_3 = 6\mathbf{e}_1 - (\cos t)\mathbf{e}_2 + e^t\mathbf{e}_3$$

Example 2.21.

$\mathbf{x} = a(\cos t)\mathbf{e}_1 + a(\sin t)\mathbf{e}_2$ traces the circle of radius a about the origin, as shown in Fig. 2-10. The derivative $\mathbf{x}' = d\mathbf{x}/dt = -a(\sin t)\mathbf{e}_1 + a(\cos t)\mathbf{e}_2$ is tangent to the circle at $\mathbf{x}$ and, as we expect, orthogonal to $\mathbf{x}$, since $\mathbf{x} \cdot \mathbf{x}' = 0$.

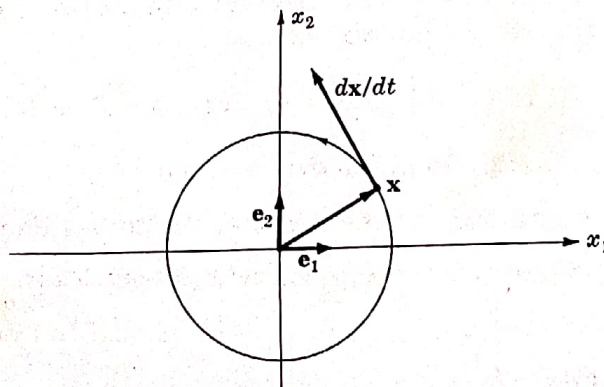


Fig. 2-10

Many properties of scalar functions carry over to vectors. For example, in Problem 2.26 we prove

Theorem 2.6. If $\mathbf{f}(t)$ is differentiable at t_0 , then $\mathbf{f}(t)$ is continuous at t_0 .

DIFFERENTIATION FORMULAS

If $\mathbf{u}$, $\mathbf{v}$, h are differentiable functions of t on I , then

$$[\mathbf{J}_1] \quad \mathbf{u} + \mathbf{v} \text{ is differentiable on } I \text{ and } \frac{d}{dt}(\mathbf{u} + \mathbf{v}) = \frac{d\mathbf{u}}{dt} + \frac{d\mathbf{v}}{dt} \quad \text{Sum rule}$$

$$[\mathbf{J}_2] \quad h\mathbf{u} \text{ is differentiable on } I \text{ and } \frac{d}{dt}(h\mathbf{u}) = h \frac{d\mathbf{u}}{dt} + \frac{dh}{dt} \mathbf{u} \quad \text{Product rule}$$

$$[\mathbf{J}_3] \quad \mathbf{u} \cdot \mathbf{v} \text{ is differentiable on } I \text{ and } \frac{d}{dt}(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot \frac{d\mathbf{v}}{dt} + \frac{d\mathbf{u}}{dt} \cdot \mathbf{v} \quad \text{Product rule}$$

$$[\mathbf{J}_4] \quad \mathbf{u} \times \mathbf{v} \text{ is differentiable on } I \text{ and } \frac{d}{dt}(\mathbf{u} \times \mathbf{v}) = \mathbf{u} \times \frac{d\mathbf{v}}{dt} + \frac{d\mathbf{u}}{dt} \times \mathbf{v}$$

Finally we have the *chain rule*:

$[\mathbf{J}_5]$ If $\mathbf{u} = \mathbf{f}(t)$ is differentiable on I_t and $t = h(\theta)$ is differentiable on I_θ , where the image $h(I_\theta)$ is contained in I_t , then $\mathbf{u} = \mathbf{g}(\theta) = \mathbf{f}(h(\theta))$ is differentiable on I_θ and

$$\frac{d\mathbf{u}}{d\theta} = \frac{d\mathbf{u}}{dt} \frac{dt}{d\theta}$$

Example 2.22.

Let $\mathbf{u} = a(\cos t)\mathbf{e}_1 - a(\sin t)\mathbf{e}_2$, $\theta = (1 + t^2)^{1/2}$, $t > 0$. Then

$$\begin{aligned} \frac{d\mathbf{u}}{d\theta} &= \frac{d\mathbf{u}}{dt} \frac{dt}{d\theta} = \frac{d\mathbf{u}}{dt} \frac{d\theta}{dt} = \frac{(-a(\sin t)\mathbf{e}_1 - a(\cos t)\mathbf{e}_2)/[t(1+t^2)^{-1/2}]}{(a/t)(1+t^2)^{1/2}} \\ &= -(\sin t)\mathbf{e}_1 - (\cos t)\mathbf{e}_2 \end{aligned}$$

where we used the fact that for scalar functions $\theta = h(t)$ such that $d\theta/dt \neq 0$, we have $dt/d\theta = 1/(d\theta/dt)$.

Example 2.23.

$$\frac{d}{dt} \left(\mathbf{u} \cdot \frac{d\mathbf{u}}{dt} \right) = \mathbf{u} \cdot \frac{d}{dt} \left(\frac{d\mathbf{u}}{dt} \right) + \frac{d\mathbf{u}}{dt} \cdot \frac{d\mathbf{u}}{dt} = \mathbf{u} \cdot \frac{d^2\mathbf{u}}{dt^2} + \left| \frac{d\mathbf{u}}{dt} \right|^2$$

Example 2.24.

$$\begin{aligned} \frac{d}{dt} \left[\mathbf{u} \cdot \frac{d\mathbf{u}}{dt} \cdot \frac{d^2\mathbf{u}}{dt^2} \right] &= \frac{d}{dt} \left(\mathbf{u} \cdot \frac{d\mathbf{u}}{dt} \times \frac{d^2\mathbf{u}}{dt^2} \right) = \mathbf{u} \cdot \frac{d}{dt} \left(\frac{d\mathbf{u}}{dt} \times \frac{d^2\mathbf{u}}{dt^2} \right) + \frac{d\mathbf{u}}{dt} \cdot \left(\frac{d\mathbf{u}}{dt} \times \frac{d^2\mathbf{u}}{dt^2} \right) \\ &= \mathbf{u} \cdot \left[\left(\frac{d\mathbf{u}}{dt} \times \frac{d^3\mathbf{u}}{dt^3} \right) + \left(\frac{d^2\mathbf{u}}{dt^2} \times \frac{d^2\mathbf{u}}{dt^2} \right) \right] + 0 = \mathbf{u} \cdot \left(\frac{d\mathbf{u}}{dt} \times \frac{d^3\mathbf{u}}{dt^3} \right) = \left[\mathbf{u} \cdot \frac{d\mathbf{u}}{dt} \cdot \frac{d^3\mathbf{u}}{dt^3} \right] \end{aligned}$$

Finally, if $\mathbf{u}$ is a vector function of constant magnitude, i.e. if $|\mathbf{u}| = \text{constant}$, then $\mathbf{u} \cdot \mathbf{u} = \text{constant}$, and, differentiating, we obtain

$$\mathbf{u} \cdot \frac{d\mathbf{u}}{dt} + \frac{d\mathbf{u}}{dt} \cdot \mathbf{u} = 0 \quad \text{or} \quad \mathbf{u} \cdot \frac{d\mathbf{u}}{dt} = 0$$

Hence $\mathbf{u}$ is orthogonal to $d\mathbf{u}/dt$. In particular we have

Theorem 2.7. If $\mathbf{u}$ is a unit vector function, then $d\mathbf{u}/dt$ is orthogonal to $\mathbf{u}$.

This theorem is an important result which will be used often.

FUNCTIONS OF CLASS C^m

In general we require that our functions can be differentiated at least once and usually twice, or more often. Also we will want to know the largest class of functions for which a result will be valid. Accordingly we say a scalar or vector valued function $\mathbf{f}$ belongs to class C^m on an interval I if the m th order derivative of $\mathbf{f}$ exists and is continuous on I . We denote the class of continuous functions by C^0 and the class of functions which have derivatives of all orders by C^∞ .

2.12. Show that the points $P(t^2, -t, 2t)$ lie in $S_{1/3}(1, -1, 2)$ for all t in $S_{1/10}(1)$.

If t is in $S_{1/10}(1)$, then $|t-1| < 1/10$ or $(t-1)^2 < 1/100$. Also,

$$\begin{aligned}(t+1)^2 &= ((t-1)+2)^2 = (t-1)^2 + 4(t-1) + 4 \\ &\leq (t-1)^2 + 4|t-1| + 4 \leq (1/100) + (4/10) + 4 \leq 5\end{aligned}$$

Now, for these t , the distance between $P(t^2, -t, 2t)$ and $(1, -1, 2)$ is

$$\begin{aligned}[(t^2-1)^2 + (-t+1)^2 + (2t-2)^2]^{1/2} &= [(t+1)^2(t-1)^2 + (t-1)^2 + 4(t-1)^2]^{1/2} \\ &\leq [(5/100) + (1/100) + (4/100)]^{1/2} \leq 1/\sqrt{10} < 1/3\end{aligned}$$

Thus P is in $S_{1/3}(1, -1, 2)$ for all t in $S_{1/10}(1)$.

2.13. Find a $\delta > 0$ such that the vectors $\mathbf{x} = t^2\mathbf{e}_1 - t\mathbf{e}_2 + 2t\mathbf{e}_3$ are in $S_{1/100}(\mathbf{e}_1 - \mathbf{e}_2 + 2\mathbf{e}_3)$ for all t in $S_\delta(1)$.

Let $\mathbf{a} = \mathbf{e}_1 - \mathbf{e}_2 + 2\mathbf{e}_3$. Then

$$\begin{aligned}|\mathbf{x} - \mathbf{a}| &= |(t^2-1)\mathbf{e}_1 - (t-1)\mathbf{e}_2 + (2t-2)\mathbf{e}_3| \\ &\leq |t^2-1||\mathbf{e}_1| + |t-1||\mathbf{e}_2| + |2t-2||\mathbf{e}_3| \leq |t-1||t+1| + |t-1| + 2|t-1| \\ &\leq |t-1|(|t+1| + 3) = |t-1|(|t-1| + 2|1| + 3) \leq |t-1|(|t-1| + 5)\end{aligned}$$

Now suppose $|t-1| < 1$; then $|\mathbf{x} - \mathbf{a}| \leq |t-1|6 < 1/100$ if $|t-1| < 1/600$. Thus if $|t-1| < 1/600$, i.e. if t is in $S_\delta(1)$ where $\delta = 1/600$, then certainly $|t-1| < 1$ and

$$|\mathbf{x} - \mathbf{a}| \leq |t-1|(|t-1| + 5) \leq |t-1|6 < (1/600)6 = 1/100$$

that is, $\mathbf{x}$ is in $S_{1/100}(\mathbf{a})$, which is the required result.

LIMITS AND CONTINUITY

2.14. Evaluate $\lim_{t \rightarrow 2} [(3t^2+1)\mathbf{e}_1 - t^3\mathbf{e}_2 + \mathbf{e}_3]$.

$$\lim_{t \rightarrow 2} [(3t^2+1)\mathbf{e}_1 - t^3\mathbf{e}_2 + \mathbf{e}_3] = \left(\lim_{t \rightarrow 2} (3t^2+1) \right) \mathbf{e}_1 - \left(\lim_{t \rightarrow 2} t^3 \right) \mathbf{e}_2 + \left(\lim_{t \rightarrow 2} 1 \right) \mathbf{e}_3 = 13\mathbf{e}_1 - 8\mathbf{e}_2 + \mathbf{e}_3$$

2.15. Let $\mathbf{f}(t) = (\sin t)\mathbf{e}_1 + t\mathbf{e}_3$ and $\mathbf{g}(t) = (t^2+1)\mathbf{e}_1 + e^t\mathbf{e}_2$. Find (a) $\lim_{t \rightarrow 0} (\mathbf{f}(t) \cdot \mathbf{g}(t))$,
(b) $\lim_{t \rightarrow 0} (\mathbf{f}(t) \times \mathbf{g}(t))$.

$$\begin{aligned}\text{(a)} \quad \lim_{t \rightarrow 0} (\mathbf{f}(t) \cdot \mathbf{g}(t)) &= \lim_{t \rightarrow 0} \mathbf{f}(t) \cdot \lim_{t \rightarrow 0} \mathbf{g}(t) \\ &= \lim_{t \rightarrow 0} ((\sin t)\mathbf{e}_1 + t\mathbf{e}_3) \cdot \lim_{t \rightarrow 0} ((t^2+1)\mathbf{e}_1 + e^t\mathbf{e}_2) \\ &= 0 \cdot (\mathbf{e}_1 + \mathbf{e}_2) = 0\end{aligned}$$

$$\begin{aligned}\text{(b)} \quad \lim_{t \rightarrow 0} (\mathbf{f}(t) \times \mathbf{g}(t)) &= \lim_{t \rightarrow 0} \mathbf{f}(t) \times \lim_{t \rightarrow 0} \mathbf{g}(t) \\ &= \lim_{t \rightarrow 0} ((\sin t)\mathbf{e}_1 + t\mathbf{e}_3) \times \lim_{t \rightarrow 0} ((t^2+1)\mathbf{e}_1 + e^t\mathbf{e}_2) \\ &= 0 \times (\mathbf{e}_1 + \mathbf{e}_2) = 0\end{aligned}$$

2.16. Define the function $\mathbf{f}(t) = \frac{\sin t}{t}\mathbf{e}_1 + (\cos t)\mathbf{e}_2$ at $t=0$ so that $\mathbf{f}(t)$ is continuous at $t=0$.

$$\lim_{t \rightarrow 0} \mathbf{f}(t) = \lim_{t \rightarrow 0} \left(\frac{\sin t}{t}\mathbf{e}_1 + (\cos t)\mathbf{e}_2 \right) = \mathbf{e}_1 + \mathbf{e}_2$$

Thus if we define $\mathbf{f}(0) = \mathbf{e}_1 + \mathbf{e}_2$, then $\lim_{t \rightarrow 0} \mathbf{f}(t) = \mathbf{f}(0)$ and $\mathbf{f}(t)$ will be continuous at $t=0$.

2.17. Prove that if $\mathbf{f}(t)$, $\mathbf{g}(t)$ and $\mathbf{h}(t)$ are continuous at t_0 , then $[\mathbf{f}(t) \mathbf{g}(t) \mathbf{h}(t)]$ is continuous at t_0 .

It is given that $\lim_{t \rightarrow t_0} \mathbf{f}(t) = \mathbf{f}(t_0)$, $\lim_{t \rightarrow t_0} \mathbf{g}(t) = \mathbf{g}(t_0)$, and $\lim_{t \rightarrow t_0} \mathbf{h}(t) = \mathbf{h}(t_0)$. It follows from Example 2.15 that

$$\lim_{t \rightarrow t_0} [\mathbf{f}(t) \mathbf{g}(t) \mathbf{h}(t)] = [\mathbf{f}(t_0) \mathbf{g}(t_0) \mathbf{h}(t_0)]$$

Hence $[\mathbf{f}(t) \mathbf{g}(t) \mathbf{h}(t)]$ is continuous at t_0 .

2.18. Using the definition of the limit, show that

$$\lim_{t \rightarrow 1} (t^2 \mathbf{e}_1 + (t+1)\mathbf{e}_2) = \mathbf{e}_1 + 2\mathbf{e}_2$$

Let $\mathbf{f}(t) = t^2 \mathbf{e}_1 + (t+1)\mathbf{e}_2$ and $\mathbf{L} = \mathbf{e}_1 + 2\mathbf{e}_2$, and consider

$$\begin{aligned} |\mathbf{f}(t) - \mathbf{L}| &= |(t^2-1)\mathbf{e}_1 + (t-1)\mathbf{e}_2| \leq |t^2-1||\mathbf{e}_1| + |t-1||\mathbf{e}_2| \\ &\leq |t-1||t+1| + |t-1| \leq |t-1|(|t-1| + 2 + 1) \leq |t-1|(|t-1| + 3). \end{aligned}$$

If we take $|t-1| < 1$, it follows further that

$$|\mathbf{f}(t) - \mathbf{L}| \leq |t-1|4 < \epsilon \quad \text{if} \quad |t-1| < \epsilon/4$$

Thus given an arbitrary $\epsilon > 0$, we are led to choose $\delta = \min(1, \epsilon/4)$. Then if $|t-1| < \delta$, i.e. if t is in $S_\delta(1)$, we have both $|t-1| < 1$ and $|t-1| < \epsilon/4$; and for these t ,

$$|\mathbf{f}(t) - \mathbf{L}| \leq |t-1|(|t-1| + 3) = |t-1|4 < \epsilon$$

i.e. $\mathbf{f}(t)$ is in $S_\epsilon(\mathbf{L})$, which is the required result.

2.19. If $\mathbf{f}(t)$ is bounded at t_0 and $\mathbf{g}(t) \rightarrow 0$ as $t \rightarrow t_0$, show that $\mathbf{f}(t) \times \mathbf{g}(t) \rightarrow 0$ as $t \rightarrow t_0$.

Let $\epsilon > 0$ be arbitrary. Since $\mathbf{f}(t)$ is bounded at t_0 , there exist $M > 0$ and $\delta_1 > 0$ such that $|\mathbf{f}(t)| \leq M$ for all $0 < |t-t_0| < \delta_1$. Also, since $\mathbf{g}(t) \rightarrow 0$ as $t \rightarrow t_0$, there exists $\delta_2 > 0$ such that $|\mathbf{g}(t)| < \epsilon/M$ for $0 < |t-t_0| < \delta_2$. Now choose $\delta = \min(\delta_1, \delta_2)$; then for $0 < |t-t_0| < \delta$ we have $0 < |t-t_0| < \delta_1$ and $0 < |t-t_0| < \delta_2$. Hence

$$|\mathbf{f}(t) \times \mathbf{g}(t) - 0| = |\mathbf{f}(t) \times \mathbf{g}(t)| = |\mathbf{f}(t)| |\mathbf{g}(t)| \sin \angle(\mathbf{f}, \mathbf{g}) \leq |\mathbf{f}(t)| |\mathbf{g}(t)| < M(\epsilon/M) = \epsilon$$

Thus $\mathbf{f}(t) \times \mathbf{g}(t) \rightarrow 0$ as $t \rightarrow t_0$.

2.20. If $\mathbf{f}(t) \rightarrow \mathbf{L}$ and $\mathbf{g}(t) \rightarrow \mathbf{M}$ as $t \rightarrow t_0$, show that $\mathbf{f}(t) \times \mathbf{g}(t) \rightarrow \mathbf{L} \times \mathbf{M}$ as $t \rightarrow t_0$.

Now let $\epsilon > 0$ be arbitrary. Since $\mathbf{g}(t) \rightarrow \mathbf{M}$ as $t \rightarrow t_0$, there exists a $\delta_1 > 0$ such that $|\mathbf{g}(t) - \mathbf{M}| < \epsilon/(2|\mathbf{L}|)$ for $0 < |t-t_0| < \delta_1$. Also $\mathbf{g}(t)$ is bounded at t_0 , and so there exist $\delta_2 > 0$ and $K > 0$ such that $|\mathbf{g}(t)| \leq K$ for $0 < |t-t_0| < \delta_2$. Finally, since $\mathbf{f}(t) \rightarrow \mathbf{L}$ as $t \rightarrow t_0$ there exists a $\delta_3 > 0$ such that $|\mathbf{f}(t) - \mathbf{L}| < \epsilon/2K$ whenever $0 < |t-t_0| < \delta_3$. Then if $0 < |t-t_0| < \delta = \min(\delta_1, \delta_2, \delta_3)$, we have $0 < |t-t_0| < \delta_1$, $0 < |t-t_0| < \delta_2$ and $0 < |t-t_0| < \delta_3$ and so

$$|(\mathbf{f} \times \mathbf{g}) - (\mathbf{L} \times \mathbf{M})| \leq |\mathbf{f} - \mathbf{L}| |\mathbf{g}| + |\mathbf{L}| |\mathbf{g} - \mathbf{M}| < (\epsilon/2K)(K) + |\mathbf{L}|(\epsilon/2|\mathbf{L}|) = \epsilon$$

DIFFERENTIATION

2.21. Let $\mathbf{u} = a(\cos t)\mathbf{e}_1 + a(\sin t)\mathbf{e}_2 + bte_3$, $a, b \neq 0$. Find (a) $\frac{d\mathbf{u}}{dt}$, (b) $\left| \frac{d\mathbf{u}}{dt} \right|$, (c) $\frac{d^2\mathbf{u}}{dt^2}$, (d) $\left| \frac{d^2\mathbf{u}}{dt^2} \right|$.

$$(a) \quad \frac{d\mathbf{u}}{dt} = \frac{d}{dt} a(\cos t)\mathbf{e}_1 + \frac{d}{dt} a(\sin t)\mathbf{e}_2 + \frac{d}{dt} (bt)\mathbf{e}_3 = -a(\sin t)\mathbf{e}_1 + a(\cos t)\mathbf{e}_2 + b\mathbf{e}_3$$

$$(b) \quad \left| \frac{d\mathbf{u}}{dt} \right| = (a^2 \sin^2 t + a^2 \cos^2 t + b^2)^{1/2} = (a^2 + b^2)^{1/2}$$

$$(c) \quad \frac{d^2\mathbf{u}}{dt^2} = \frac{d}{dt} \left(\frac{d\mathbf{u}}{dt} \right) = \frac{d}{dt} (-a(\sin t)\mathbf{e}_1 + a(\cos t)\mathbf{e}_2 + b\mathbf{e}_3) = -a(\cos t)\mathbf{e}_1 - a(\sin t)\mathbf{e}_2$$

$$(d) \quad \left| \frac{d^2\mathbf{u}}{dt^2} \right| = (a^2 \cos^2 t + a^2 \sin^2 t)^{1/2} = |a|$$

2.22. Find the equation of the line tangent to the curve generated by $\mathbf{x} = t\mathbf{e}_1 + t^2\mathbf{e}_2 + t^3\mathbf{e}_3$ at $t = 1$.

A vector tangent to the curve at the point $\mathbf{x}$ is $d\mathbf{x}/dt = \mathbf{e}_1 + 2t\mathbf{e}_2 + 3t^2\mathbf{e}_3$. If $\mathbf{y}$ denotes a generic point on the tangent line, then the equation of the tangent line (see Fig. 2-15) is

$$(\mathbf{y} - \mathbf{x}) = k \frac{d\mathbf{x}}{dt} \quad \text{or} \quad \mathbf{y} = k \frac{d\mathbf{x}}{dt} + \mathbf{x}, \quad -\infty < k < \infty$$

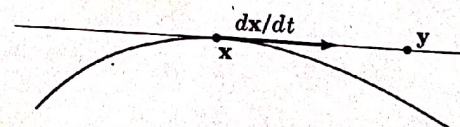


Fig. 2-15

At $t = 1$, $\mathbf{x} = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$ and $d\mathbf{x}/dt = \mathbf{e}_1 + 2\mathbf{e}_2 + 3\mathbf{e}_3$. Hence the tangent line at $t = 1$ is

$$\mathbf{y} = k(\mathbf{e}_1 + 2\mathbf{e}_2 + 3\mathbf{e}_3) + (\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3), \quad -\infty < k < \infty$$

or

$$\mathbf{y} = (k+1)\mathbf{e}_1 + (2k+1)\mathbf{e}_2 + (3k+1)\mathbf{e}_3, \quad -\infty < k < \infty$$

2.23. If $\mathbf{u} = (3t^2 + 1)\mathbf{e}_1 + (\sin t)\mathbf{e}_2$ and $\mathbf{v} = (\cos t)\mathbf{e}_1 + e^t\mathbf{e}_3$, find (a) $\frac{d}{dt}(\mathbf{u} \cdot \mathbf{v})$, (b) $\frac{d}{dt}(\mathbf{u} \times \mathbf{v})$, (c) $\frac{d}{dt}|\mathbf{u}|$.

$$\begin{aligned} (a) \quad \frac{d}{dt}(\mathbf{u} \cdot \mathbf{v}) &= \mathbf{u} \cdot \frac{d\mathbf{v}}{dt} + \frac{d\mathbf{u}}{dt} \cdot \mathbf{v} = ((3t^2 + 1)\mathbf{e}_1 + (\sin t)\mathbf{e}_2) \cdot (-\sin t)\mathbf{e}_1 + e^t\mathbf{e}_3 \\ &\quad + (6t\mathbf{e}_1 + (\cos t)\mathbf{e}_2) \cdot ((\cos t)\mathbf{e}_1 + e^t\mathbf{e}_3) \\ &= -(3t^2 + 1)\sin t + 6t\cos t \end{aligned}$$

$$\begin{aligned} (b) \quad \frac{d}{dt}(\mathbf{u} \times \mathbf{v}) &= \mathbf{u} \times \frac{d\mathbf{v}}{dt} + \frac{d\mathbf{u}}{dt} \times \mathbf{v} \\ &= \det \begin{pmatrix} \mathbf{e}_1 & 3t^2 + 1 & -\sin t \\ \mathbf{e}_2 & \sin t & 0 \\ \mathbf{e}_3 & 0 & e^t \end{pmatrix} + \det \begin{pmatrix} \mathbf{e}_1 & 6t & \cos t \\ \mathbf{e}_2 & \cos t & 0 \\ \mathbf{e}_3 & 0 & e^t \end{pmatrix} \\ &= (\sin t)e^t\mathbf{e}_1 - (3t^2 + 1)e^t\mathbf{e}_2 + (\sin^2 t)\mathbf{e}_3 + (\cos t)e^t\mathbf{e}_1 - 6te^t\mathbf{e}_2 - (\cos^2 t)\mathbf{e}_3 \\ &= (\sin t + \cos t)e^t\mathbf{e}_1 - (3t^2 + 6t + 1)e^t\mathbf{e}_2 + (\sin^2 t - \cos^2 t)\mathbf{e}_3 \end{aligned}$$

Another method:

$$\mathbf{u} \times \mathbf{v} = \det \begin{pmatrix} \mathbf{e}_1 & 3t^2 + 1 & \cos t \\ \mathbf{e}_2 & \sin t & 0 \\ \mathbf{e}_3 & 0 & e^t \end{pmatrix} = (\sin t)e^t\mathbf{e}_1 - (3t^2 + 1)e^t\mathbf{e}_2 - (\sin t \cos t)\mathbf{e}_3$$

and

$$\begin{aligned} \frac{d}{dt}(\mathbf{u} \times \mathbf{v}) &= [(\sin t)e^t + (\cos t)e^t]\mathbf{e}_1 - [(3t^2 + 1)e^t + 6te^t]\mathbf{e}_2 - [-\sin^2 t + \cos^2 t]\mathbf{e}_3 \\ &= (\sin t + \cos t)e^t\mathbf{e}_1 - (3t^2 + 6t + 1)e^t\mathbf{e}_2 + (\sin^2 t - \cos^2 t)\mathbf{e}_3 \end{aligned}$$

$$\begin{aligned} (c) \quad \frac{d}{dt}|\mathbf{u}| &= \frac{d}{dt}(\mathbf{u} \cdot \mathbf{u})^{1/2} = \frac{1}{2}(\mathbf{u} \cdot \mathbf{u})^{-1/2} \frac{d}{dt}(\mathbf{u} \cdot \mathbf{u}) = \frac{1}{2}(\mathbf{u} \cdot \mathbf{u})^{-1/2} \left(\mathbf{u} \cdot \frac{d\mathbf{u}}{dt} \right) = \frac{(\mathbf{u} \cdot \frac{d\mathbf{u}}{dt})}{|\mathbf{u}|} \\ &= [(3t^2 + 1)\mathbf{e}_1 + (\sin t)\mathbf{e}_2] \cdot [(3t^2 + 1)\mathbf{e}_1 + (\sin t)\mathbf{e}_2] / [(3t^2 + 1)^2 + \sin^2 t]^{1/2} \\ &= (18t^3 + 6t + \sin t \cos t) / [(3t^2 + 1)^2 + \sin^2 t]^{1/2} \end{aligned}$$

$\theta = \log t$

2.24. Let $\mathbf{u} = (\sin t)\mathbf{e}_1 + 2t^2\mathbf{e}_2 + t\mathbf{e}_3$, ($t > 0$), and $t = \log \theta$. Find $d\mathbf{u}/d\theta$ as a (a) function of θ , (b) function of t .

$$\begin{aligned} (a) \quad \frac{d\mathbf{u}}{d\theta} &= \frac{d\mathbf{u}}{dt} \frac{dt}{d\theta} = ((\cos t)\mathbf{e}_1 + 4t\mathbf{e}_2 + \mathbf{e}_3)(1/\theta). \quad \text{Substituting for } t, \\ &\quad d\mathbf{u}/d\theta = (1/\theta)((\cos \log \theta)\mathbf{e}_1 + 4(\log \theta)\mathbf{e}_2 + \mathbf{e}_3). \end{aligned}$$

Another method:

Substituting for t , we have $\mathbf{u} = (\sin \log \theta)\mathbf{e}_1 + 2(\log^2 \theta)\mathbf{e}_2 + (\log \theta)\mathbf{e}_3$. Hence

$$\begin{aligned} \frac{d\mathbf{u}}{d\theta} &= (\cos \log \theta)(1/\theta)\mathbf{e}_1 + 4(\log \theta)(1/\theta)\mathbf{e}_2 + (1/\theta)\mathbf{e}_3 \\ &= (1/\theta)((\cos \log \theta)\mathbf{e}_1 + 4(\log \theta)\mathbf{e}_2 + \mathbf{e}_3) \end{aligned}$$

$$(b) \quad \frac{d\mathbf{u}}{d\theta} = \frac{d\mathbf{u}}{dt} \frac{dt}{d\theta} = ((\cos t)\mathbf{e}_1 + 4t\mathbf{e}_2 + \mathbf{e}_3)(1/\theta) = e^{-t}((\cos t)\mathbf{e}_1 + 4t\mathbf{e}_2 + \mathbf{e}_3)$$

Another method:

Since $\theta = e^t$, $d\theta/dt = e^t$. Hence

$$\frac{d\mathbf{u}}{d\theta} = \frac{d\mathbf{u}}{dt} \frac{dt}{d\theta} = \frac{d\mathbf{u}}{dt} / \frac{d\theta}{dt} = e^{-t}((\cos t)\mathbf{e}_1 + 4t\mathbf{e}_2 + \mathbf{e}_3)$$

2.25. Using the definition of the derivative, show that:

(a) If $f(t) = a$, $a = \text{constant}$, then $f'(t) = 0$.

(b) If $f(t) = ah(t)$, $a = \text{constant}$, then $f'(t) = ah'(t)$.

$$(a) \quad f'(t) = \lim_{\Delta t \rightarrow 0} \frac{f(t + \Delta t) - f(t)}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{a - a}{\Delta t} = \lim_{\Delta t \rightarrow 0} 0 = 0$$

$$(b) \quad f'(t) = \lim_{\Delta t \rightarrow 0} \frac{f(t + \Delta t) - f(t)}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{ah(t + \Delta t) - ah(t)}{\Delta t} \\ = \lim_{\Delta t \rightarrow 0} a \lim_{\Delta t \rightarrow 0} \frac{h(t + \Delta t) - h(t)}{\Delta t} = ah'(t)$$

2.26. Prove Theorem 2.6: If $f(t)$ is differentiable at t_0 , then $f(t)$ is continuous at t_0 .

We consider

$$\lim_{t \rightarrow t_0} [f(t) - f(t_0)] = \lim_{t \rightarrow t_0} \frac{f(t) - f(t_0)}{t - t_0} (t - t_0) = \lim_{t \rightarrow t_0} \frac{f(t) - f(t_0)}{t - t_0} \lim_{t \rightarrow t_0} (t - t_0) = [f'(t_0)] 0 = 0$$

and so $f(t)$ is continuous at t_0 .

2.27. If u and v are differentiable functions of t , show that

$$\frac{d}{dt}(u + v) = \frac{du}{dt} + \frac{dv}{dt}$$

We write $w(t) = u(t) + v(t)$. Then

$$\begin{aligned} \frac{d}{dt}(u + v) &= \frac{dw}{dt} = \lim_{\Delta t \rightarrow 0} \frac{w(t + \Delta t) - w(t)}{\Delta t} \\ &= \lim_{\Delta t \rightarrow 0} \frac{u(t + \Delta t) + v(t + \Delta t) - u(t) - v(t)}{\Delta t} \\ &= \lim_{\Delta t \rightarrow 0} \frac{u(t + \Delta t) - u(t)}{\Delta t} + \lim_{\Delta t \rightarrow 0} \frac{v(t + \Delta t) - v(t)}{\Delta t} = \frac{du}{dt} + \frac{dv}{dt} \end{aligned}$$

2.28. If u and v are differentiable functions of t , show that $\frac{d}{dt}(u \times v) = u \times \frac{dv}{dt} + \frac{du}{dt} \times v$.

Let $w(t) = u(t) \times v(t)$. Then

$$\begin{aligned} \frac{d}{dt}(u \times v) &= \frac{dw}{dt} = \lim_{\Delta t \rightarrow 0} \frac{w(t + \Delta t) - w(t)}{\Delta t} \\ &= \lim_{\Delta t \rightarrow 0} \frac{u(t + \Delta t) \times v(t + \Delta t) - u(t) \times v(t)}{\Delta t} \\ &= \lim_{\Delta t \rightarrow 0} \left[\frac{u(t + \Delta t) \times (v(t + \Delta t) - v(t))}{\Delta t} + \frac{(u(t + \Delta t) - u(t)) \times v(t)}{\Delta t} \right] \\ &= \lim_{\Delta t \rightarrow 0} u(t + \Delta t) \times \lim_{\Delta t \rightarrow 0} \frac{v(t + \Delta t) - v(t)}{\Delta t} + \lim_{\Delta t \rightarrow 0} \frac{u(t + \Delta t) - u(t)}{\Delta t} \times \lim_{\Delta t \rightarrow 0} v(t) \\ &= u \times \frac{dv}{dt} + \frac{du}{dt} \times v \end{aligned}$$

where we used the fact that $u(t)$, since it is differentiable, is continuous. Then $\lim_{\Delta t \rightarrow 0} u(t + \Delta t) = u(t)$, and $v(t)$ is independent of Δt ; hence $\lim_{\Delta t \rightarrow 0} v(t) = v(t)$. Another method is to find u and v in terms of a basis and to differentiate componentwise.

2.29. Show that $\mathbf{u} = \mathbf{a} \cos kt + \mathbf{b} \sin kt$, $\mathbf{a}, \mathbf{b} = \text{constant}$, is a solution to $d^2\mathbf{u}/dt^2 = -k^2\mathbf{u}$.

$$\begin{aligned} d\mathbf{u}/dt &= \mathbf{a} \frac{d}{dt} \cos kt + \mathbf{b} \frac{d}{dt} \sin kt = -k\mathbf{a} \sin kt + k\mathbf{b} \cos kt \\ d^2\mathbf{u}/dt^2 &= -k^2\mathbf{a} \cos kt - k^2\mathbf{b} \sin kt = -k^2(\mathbf{a} \cos kt + \mathbf{b} \sin kt) = -k^2\mathbf{u} \end{aligned}$$

2.30. Show that $\mathbf{u} \cdot \frac{d\mathbf{u}}{dt} = |\mathbf{u}| \frac{d|\mathbf{u}|}{dt}$.

$$\frac{d}{dt}(\mathbf{u} \cdot \mathbf{u}) = \frac{d}{dt}|\mathbf{u}|^2, \quad \mathbf{u} \cdot \frac{d\mathbf{u}}{dt} + \frac{d\mathbf{u}}{dt} \cdot \mathbf{u} = 2\mathbf{u} \cdot \frac{d\mathbf{u}}{dt} = 2|\mathbf{u}| \frac{d|\mathbf{u}|}{dt}, \quad \text{or} \quad \mathbf{u} \cdot \frac{d\mathbf{u}}{dt} = |\mathbf{u}| \frac{d|\mathbf{u}|}{dt}$$

2.31. If $\mathbf{f}(t)$ is differentiable at t_0 , prove that

$$\mathbf{f}(t_0 + \Delta t) = \mathbf{f}(t_0) + \mathbf{f}'(t_0)\Delta t + \mathbf{R}(t_0, \Delta t)$$

where $(\mathbf{R}(t_0, \Delta t)/\Delta t) \rightarrow 0$ as $\Delta t \rightarrow 0$.

Define $\mathbf{R} = \mathbf{f}(t_0 + \Delta t) - \mathbf{f}(t_0) - \mathbf{f}'(t_0)\Delta t$. Then

$$\begin{aligned} \lim_{\Delta t \rightarrow 0} \mathbf{R}/\Delta t &= \lim_{\Delta t \rightarrow 0} [\mathbf{f}(t_0 + \Delta t) - \mathbf{f}(t_0) - \mathbf{f}'(t_0)\Delta t]/\Delta t \\ &= \lim_{\Delta t \rightarrow 0} \left[\frac{\mathbf{f}(t_0 + \Delta t) - \mathbf{f}(t_0)}{\Delta t} - \mathbf{f}'(t_0) \right] = \mathbf{f}'(t_0) - \mathbf{f}'(t_0) = 0 \end{aligned}$$

which is the required result.

2.32. Prove the converse of Problem 2.31. That is, if there exists a linear function $\mathbf{a}\Delta t$ of Δt , such that $\mathbf{f}(t_0 + \Delta t) = \mathbf{f}(t_0) + \mathbf{a}\Delta t + \mathbf{R}$, where $\lim_{\Delta t \rightarrow 0} \mathbf{R}/\Delta t = 0$, then $\mathbf{f}(t)$ is differentiable at t_0 and $\mathbf{a} = \mathbf{f}'(t_0)$.

$$\lim_{\Delta t \rightarrow 0} \frac{\mathbf{f}(t_0 + \Delta t) - \mathbf{f}(t_0)}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{\mathbf{a}\Delta t + \mathbf{R}}{\Delta t} = \lim_{\Delta t \rightarrow 0} \mathbf{a} + \lim_{\Delta t \rightarrow 0} \mathbf{R}/\Delta t = \mathbf{a}$$

Hence $\mathbf{f}(t)$ is differentiable at t_0 and $\mathbf{f}'(t_0) = \mathbf{a}$.

TAYLOR'S THEOREM AND ANALYTIC FUNCTIONS

2.33. Show that

$$\begin{aligned} (\sin t)\mathbf{e}_1 + (t^2 + 1)\mathbf{e}_2 &= \mathbf{e}_1 + \frac{1}{4}(\pi^2 + 4)\mathbf{e}_2 + \pi\mathbf{e}_2(t - \pi/2) \\ &\quad + \frac{1}{2}(-\mathbf{e}_1 + 2\mathbf{e}_2)(t - \pi/2)^2 + o[(t - \pi/2)^2] \end{aligned}$$

We find the first three terms of the Taylor expansion of $\mathbf{f}(t) = (\sin t)\mathbf{e}_1 + (t^2 + 1)\mathbf{e}_2$ about $t = \pi/2$. Namely,

$$\begin{aligned} \mathbf{f}(t) &= (\sin t)\mathbf{e}_1 + (t^2 + 1)\mathbf{e}_2 & \mathbf{f}(\pi/2) &= \mathbf{e}_1 + \frac{1}{4}(\pi^2 + 4)\mathbf{e}_2 \\ \mathbf{f}'(t) &= \cos t\mathbf{e}_1 + 2t\mathbf{e}_2 & \mathbf{f}'(\pi/2) &= \pi\mathbf{e}_2 \\ \mathbf{f}''(t) &= -\sin t\mathbf{e}_1 + 2\mathbf{e}_2 & \mathbf{f}''(\pi/2) &= -\mathbf{e}_1 + 2\mathbf{e}_2 \end{aligned}$$

Thus $(\sin t)\mathbf{e}_1 + (t^2 + 1)\mathbf{e}_2 = \mathbf{e}_1 + \frac{1}{4}(\pi^2 + 4)\mathbf{e}_2 + \pi\mathbf{e}_2(t - \pi/2) + \frac{1}{2}(-\mathbf{e}_1 + 2\mathbf{e}_2)(t - \pi/2)^2 + \mathbf{R}$

where $\lim_{t \rightarrow \pi/2} \mathbf{R}/(t - \pi/2)^2 = 0$.

2.34. Show that at $t = 0$: (a) $to(t^2) = o(t^3)$, (b) $o(t^2) + o(t^3) = o(t^2)$, (c) $o(t^2) \cdot o(t^3) = o(t^5)$.

$$(a) \quad \lim_{t \rightarrow 0} to(t^2)/t^3 = \lim_{t \rightarrow 0} (t/t)(o(t^2)/t^2) = \lim_{t \rightarrow 0} o(t^2)/t^2 = 0$$

$$(b) \quad \lim_{t \rightarrow 0} (o(t^2) + o(t^3))/t^2 = \lim_{t \rightarrow 0} o(t^2)/t^2 + \lim_{t \rightarrow 0} t(o(t^3)/t^3) = 0$$

$$(c) \quad \lim_{t \rightarrow 0} o(t^2) \cdot o(t^3)/t^5 = \lim_{t \rightarrow 0} o(t^2)/t^2 \lim_{t \rightarrow 0} o(t^3)/t^3 = 0$$

Supplementary Problems

- 2.39. Find the equation of the plane through $A(1, 0, -1)$, $B(0, 0, 1)$, $C(-1, -1, 0)$.
 Ans. $2x_1 - 3x_2 + x_3 = 1$
- 2.40. Find the equation of the plane through $A(1, -1, 0)$ and normal to the line $x_1 = -k + 1$, $x_2 = k + 1$, $x_3 = 3$.
 Ans. $x_1 - x_2 = 2$
- 2.41. Find the equations of the line which is the intersection of the planes $3x_1 - 2x_2 + x_3 = 5$ and $2x_1 + 3x_2 - x_3 = -1$.
 Ans. $x_1 = -k + 1$, $x_2 = 5k - 1$, $x_3 = 13k$, $(-\infty < k < \infty)$
- 2.42. Prove that the equation of the line through $\mathbf{a}$ and normal to the plane $\mathbf{x} \cdot \mathbf{n} = d$, $|\mathbf{n}| \neq 0$, is $\mathbf{x} = k\mathbf{n} + \mathbf{a}$, $-\infty < k < \infty$.
- 2.43. Prove that the equation of the line through $\mathbf{a}$ and orthogonal to $\mathbf{c}$ and $\mathbf{d}$, $\mathbf{c} \times \mathbf{d} \neq 0$, is $\mathbf{x} = k(\mathbf{c} \times \mathbf{d}) + \mathbf{a}$.
- 2.44. Find the equation of the cone with vertex at $A(0, 1, 1)$, with axis parallel to the x_1 axis, and half angle $\theta = 60^\circ$.
 Ans. $3x_1^2 - (x_2 - 1)^2 - (x_3 - 1)^2 = 0$
- 2.45. Calculate the vectors $\mathbf{x} = (t^3 + 1)\mathbf{e}_1 + (1 - t^2)\mathbf{e}_2$ for integer t in $-4 \leq t \leq 4$ and sketch.
- 2.46. Let $\mathbf{f}(t) = (t^2 + 1)\mathbf{e}_1 + t^3\mathbf{e}_3$ and $\mathbf{g}(t) = (\sin t)\mathbf{e}_1 - (\cos t)\mathbf{e}_2$. Find (a) $\mathbf{f}(a + b)$, (b) $\mathbf{g}(t + \Delta t)$, (c) $\mathbf{f}(\sin t) \times \mathbf{g}(t^2 + 1)$.
 Ans. (a) $(a^2 + 2ab + b^2 + 1)\mathbf{e}_1 + (a^3 + 3a^2b + 3b^2a + b^3)\mathbf{e}_3$
 (b) $\sin(t + \Delta t)\mathbf{e}_1 - \cos(t + \Delta t)\mathbf{e}_2$
 (c) $(\cos(t^2 + 1) \sin^3 t)\mathbf{e}_1 + (\sin(t^2 + 1) \sin^3 t)\mathbf{e}_2 - (\cos(t^2 + 1)(1 + \sin^2 t))\mathbf{e}_3$
- 2.47. If $\mathbf{a} = 2\mathbf{e}_1 - \mathbf{e}_2 + \mathbf{e}_3$ and $\mathbf{b} = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$, show that $\mathbf{b}$ is in $S_4(\mathbf{a})$ and find $\delta > 0$ such that $S_\delta(\mathbf{b})$ is contained in $S_4(\mathbf{a})$.
 Ans. $2\mathbf{e}_1 + (1/\epsilon)\mathbf{e}_2 - 2\mathbf{e}_3$
- 2.48. Evaluate $\lim_{t \rightarrow -1} [(t^2 + 1)\mathbf{e}_1 + t^2\mathbf{e}_2 + ((t^2 - 1)/(t + 1))\mathbf{e}_3]$.
 Ans. $2\mathbf{e}_1 + (1/\epsilon)\mathbf{e}_2 - 2\mathbf{e}_3$
- 2.49. Determine the values of t for which $\mathbf{f}(t) = [(t^2 + 1)/(t^2 - 1)]\mathbf{e}_1 + (\tan t)\mathbf{e}_2$ is discontinuous.
 Ans. $t = 1, -1, \frac{1}{2}\pi \pm n\pi$, $n = 0, 1, 2, \dots$
- 2.50. Let $\mathbf{f}(t) = (t^2 - 1)\mathbf{e}_2 + (\cos t)\mathbf{e}_3$ and $\mathbf{g}(t) = (\sin t)\mathbf{e}_1 + t^2\mathbf{e}_2$. Find (a) $\lim_{t \rightarrow 0} (\mathbf{f}(t) \cdot \mathbf{g}(t))$, (b) $\lim_{t \rightarrow 0} (\mathbf{f}(t) \times \mathbf{g}(t))$.
 Ans. (a) -1 , (b) $-\mathbf{e}_1$
- 2.51. If $\mathbf{f}(t)$, $\mathbf{g}(t)$ and $\mathbf{h}(t)$ are continuous on I , show that $\mathbf{f}(t) \times (\mathbf{g}(t) \times \mathbf{h}(t))$ is continuous on I .
- 2.52. If $\mathbf{u} = (t^2 + 1)\mathbf{e}_1 - t^2\mathbf{e}_2 + (\log t)\mathbf{e}_3$, $t > 0$, find (a) du/dt , (b) d^2u/dt^2 .
 Ans. (a) $2t\mathbf{e}_1 - (t + 1)t^2\mathbf{e}_2 + (1/t)\mathbf{e}_3$, (b) $2\mathbf{e}_1 - (t + 2)t^2\mathbf{e}_2 - (1/t^3)\mathbf{e}_3$
- 2.53. Find the equation of the line tangent to the curve traced by $\mathbf{x} = (t^2 - 2)\mathbf{e}_1 + (t + 3)\mathbf{e}_2 + (t^4 + 4t + 1)\mathbf{e}_3$ at $t = 1$.
 Ans. $\mathbf{x} = (2k - 1)\mathbf{e}_1 + (k + 4)\mathbf{e}_2 + (8k + 6)\mathbf{e}_3$, $-\infty < k < \infty$
- 2.54. If $\mathbf{u} = (2 + t)\mathbf{e}_2 + (\log t)\mathbf{e}_3$ and $\mathbf{v} = (\sin t)\mathbf{e}_1 - (\cos t)\mathbf{e}_2$, $t > 0$, find (a) $\frac{d}{dt}(\mathbf{u} \cdot \mathbf{v})$, (b) $\frac{d}{dt}(\mathbf{u} \times \mathbf{v})$.
 Ans. (a) $(2 + t) \sin t - \cos t$
 (b) $[(1/t) \cos t - \log t \sin t]\mathbf{e}_1 + [(1/t) \sin t + \log t \cos t]\mathbf{e}_2 - [(2 + t) \cos t + \sin t]\mathbf{e}_3$
- 2.55. Let $\mathbf{u} = t^2\mathbf{e}_1 + 2(\sin t)\mathbf{e}_2 + (t^2 + 1)\mathbf{e}_3$ and $t = \theta^2 + 2$, $t \geq 2$. Find $du/d\theta$ and $d^2u/d\theta^2$ as functions of t .
 Ans. $du/d\theta = 2(t - 2)^{1/2}(t^2\mathbf{e}_1 + 2(\cos t)\mathbf{e}_2 + 2t\mathbf{e}_3)$
 $d^2u/d\theta^2 = (4t - 6)t^2\mathbf{e}_1 + [4 \cos t - (8t - 16) \sin t]\mathbf{e}_2 + (12t - 16)\mathbf{e}_3$
- 2.56. Show that $\frac{d}{dt} \left(\mathbf{u} \cdot \frac{d\mathbf{v}}{dt} - \frac{d\mathbf{u}}{dt} \cdot \mathbf{v} \right) = \mathbf{u} \cdot \frac{d^2\mathbf{v}}{dt^2} - \frac{d^2\mathbf{u}}{dt^2} \cdot \mathbf{v}$.
- 2.57. Find the first three terms of the Taylor expansion of $\mathbf{f}(t) = (\cos t)\mathbf{e}_1 + (t^2 + 2t + 1)\mathbf{e}_2$ about $t = 0$.
 Ans. $(\mathbf{e}_1 + \mathbf{e}_2) + 2\mathbf{e}_2t - \mathbf{e}_1t^2/2 + t^2\mathbf{e}_2$

- 2.58. Using the definition of the derivative (Δ process), show that
- $$\frac{d}{dt}[(t^2 + 1)\mathbf{e}_1 + (1/(t + 1))\mathbf{e}_2 + \mathbf{e}_3] = 2t\mathbf{e}_1 - (1/(t + 1)^2)\mathbf{e}_2$$
- 2.59. If $\mathbf{u}$ and $\mathbf{v}$ are differentiable functions of t , show that $\frac{d}{dt}(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot \frac{d\mathbf{v}}{dt} + \frac{d\mathbf{u}}{dt} \cdot \mathbf{v}$.
- 2.60. Find all $\mathbf{u}$ such that $d\mathbf{u}/dt = (3t^2 + 1)\mathbf{e}_1 + t^3\mathbf{e}_2 - (\sin t)\mathbf{e}_3$.
 Ans. $\mathbf{u} = (t^3 + t + C_1)\mathbf{e}_1 + (t^4/4 + C_2)\mathbf{e}_2 + (\cos t + C_3)\mathbf{e}_3$
- 2.61. Find all $\mathbf{u}$ such that $d^2\mathbf{u}/dt^2 = at^2 + bt + c$, $a, b, c = \text{constants}$.
 Ans. $\mathbf{u} = \frac{1}{12}at^4 + \frac{1}{6}bt^3 + \frac{1}{2}ct^2 + C_1t + C_2$
- 2.62. Show that $\mathbf{u} \times d\mathbf{u}/dt = \mathbf{0}$ if and only if $\mathbf{u}$ has constant direction.
- 2.63. Show that $(\tan^2 t)\mathbf{e}_1 + (2t^3 + t^4)\mathbf{e}_2 = \mathbf{O}(t^2)$ at $t = 0$.
- 2.64. Let $f(t)$ be analytic at $t = t_0$ and let
 $f'(t_0) = 0, f''(t_0) = 0, \dots, f^{(n)}(t_0) = 0, f^{(n+1)}(t_0) \neq 0$
 Show that $f^{(n+1)}(t_0)$ represents a vector tangent to the curve $\mathbf{x} = f(t)$ at $f(t_0)$.
- 2.65. If $f(t) = \begin{cases} e^{-(1/t)^2} & \text{for } t \neq 0 \\ 0 & \text{for } t = 0 \end{cases}$, show that $f^{(n)}(0) = 0$ for all n .
- 2.66. Show that the vectors $\mathbf{x} = (t^2 + 1)\mathbf{e}_1 + (t + 1)\mathbf{e}_2 - t\mathbf{e}_3$ lie in $S_{1/10}(5\mathbf{e}_1 + 3\mathbf{e}_2 - 2\mathbf{e}_3)$ for all t in $S_{1/100}(2)$.
- 2.67. If $\lim_{t \rightarrow t_0} [f_1(t)\mathbf{e}_1 + f_2(t)\mathbf{e}_2 + f_3(t)\mathbf{e}_3] = L_1\mathbf{e}_1 + L_2\mathbf{e}_2 + L_3\mathbf{e}_3$, show that $\lim_{t \rightarrow t_0} f_i(t) = L_i$, $i = 1, 2, 3$.
- 2.68. If $f(t) \rightarrow \mathbf{L}$ and $g(t) \rightarrow \mathbf{M}$ as $t \rightarrow t_0$, show that $f(t) \cdot g(t) \rightarrow \mathbf{L} \cdot \mathbf{M}$ as $t \rightarrow t_0$.
- 2.69. Show that $[at^2 + o(t^3)] \cdot [bt + o(t^2)] = a \cdot bt^3 + o(t^4)$, $a, b = \text{constants}$.
- 2.70. Show that $g(t) = O(g(t))$.
- 2.71. Show that $o(g_1(t)) \cdot O(g_2(t)) = o(g_1(t)g_2(t))$.
- 2.72. Show that $O(g_1(t)) \times O(g_2(t)) = O(g_1(t)g_2(t))$.
- 2.73. If $f(t) \rightarrow f(t_0)$ as $t \rightarrow t_0$, $t = h(\theta)$, and $h(\theta) \rightarrow t_0$ as $\theta \rightarrow \theta_0$, show that $f(h(\theta)) \rightarrow f(t_0)$ as $\theta \rightarrow \theta_0$.
- 2.74. Prove the chain rule: If $\mathbf{u} = f(t)$ and $t = h(\theta)$ are differentiable functions of t and θ respectively, then $\frac{d\mathbf{u}}{d\theta} = \frac{d\mathbf{u}}{dt} \frac{dt}{d\theta}$.

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Chapter 3

Concept of a Curve

REGULAR REPRESENTATIONS

By a *regular parametric representation* we mean a vector function

$$\mathbf{x} = \mathbf{x}(t), \quad t \in I \quad (3.1)$$

of t in an interval I with the property that

(i) $\mathbf{x}(t)$ is of class C^1 in I

(ii) $\mathbf{x}'(t) \neq 0$ for all t in I

The variable t is called the *parameter* of the representation.

If a basis is chosen in (E^3) the equation $\mathbf{x} = \mathbf{x}(t)$ is equivalent to three scalar equations

$$x_1 = x_1(t), \quad x_2 = x_2(t), \quad x_3 = x_3(t), \quad t \in I \quad (3.2)$$

the components of $\mathbf{x} = \mathbf{x}(t)$ with respect to the basis. Evidently $\mathbf{x} = \mathbf{x}(t)$ is a regular parametric representation if and only if each $x_i(t)$ is in class C^1 and if for each t in I at least one of the $x_i'(t) \neq 0$.

Example 3.1.

The function $\mathbf{x} = (t+1)\mathbf{e}_1 + (t^2+3)\mathbf{e}_2, \quad -\infty < t < \infty$, is a regular parametric representation, since $\mathbf{x}' = \mathbf{e}_1 + 2t\mathbf{e}_2$ is continuous and $\mathbf{x}' \neq 0$ for all t . The image of the function is the parabola shown in Fig. 3-1.

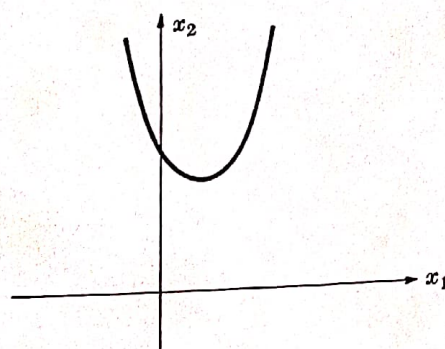


Fig. 3-1

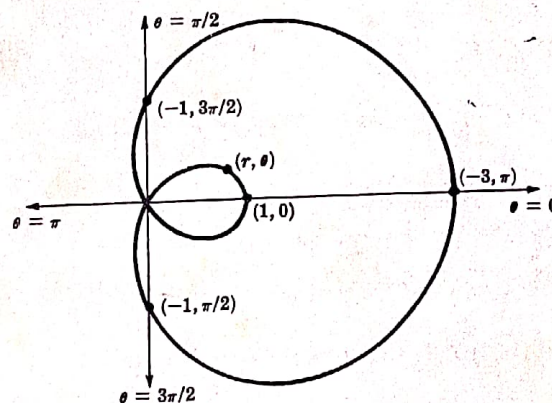


Fig. 3-2

Example 3.2.

The graph of the equation $r = 2 \cos \theta - 1, \quad 0 \leq \theta \leq 2\pi$, in polar coordinates is shown in Fig. 3-2. Polar and rectangular coordinates are related by the equations $x_1 = r \cos \theta, \quad x_2 = r \sin \theta$. Upon substitution for r , we obtain the representation

$$x_1 = (\cos \theta)(2 \cos \theta - 1), \quad x_2 = (\sin \theta)(2 \cos \theta - 1), \quad 0 \leq \theta \leq 2\pi$$

or

$$\mathbf{x} = (\cos \theta)(2 \cos \theta - 1)\mathbf{e}_1 + (\sin \theta)(2 \cos \theta - 1)\mathbf{e}_2$$

This representation is regular, since

$$\mathbf{x}' = [-4 \sin \theta \cos \theta + \sin \theta] \mathbf{e}_1 + [2 \cos^2 \theta - 2 \sin^2 \theta - \cos \theta] \mathbf{e}_2$$

is continuous and it can be computed that $|\mathbf{x}'| = \sqrt{5 - 4 \cos \theta} \neq 0$ for all θ and hence $\mathbf{x}' \neq 0$ for all θ .

A regular parametric representation $\mathbf{x} = \mathbf{x}(t)$ on I can have multiple points, i.e. $t_1 \neq t_2$ in I for which $\mathbf{x}(t_1) = \mathbf{x}(t_2)$. However, *locally* this will not be the case. In Problem 3.7, we prove

Theorem 3.1. If $\mathbf{x} = \mathbf{x}(t)$ is a regular parametric representation on I , then for each t_0 in I there exists a neighborhood of t_0 in which $\mathbf{x}(t)$ is one-to-one.

Example 3.3.

The function $\mathbf{x} = a(\cos \theta) \mathbf{e}_1 + a(\sin \theta) \mathbf{e}_2$, $a \neq 0$, $(-\infty < \theta < \infty)$ is a regular representation of the circle of radius $|a|$ about the origin, since $d\mathbf{x}/d\theta = -a(\sin \theta) \mathbf{e}_1 + a(\cos \theta) \mathbf{e}_2$ is continuous for all θ and

$$|d\mathbf{x}/d\theta| = |-a(\sin \theta) \mathbf{e}_1 + a(\cos \theta) \mathbf{e}_2| = |a| \neq 0$$

Observe that every point on this representation is a multiple point, since for any θ_0 ,

$$a \cos(\theta_0 + 2\pi) \mathbf{e}_1 + a \sin(\theta_0 + 2\pi) \mathbf{e}_2 = a(\cos \theta_0) \mathbf{e}_1 + a(\sin \theta_0) \mathbf{e}_2$$

However, restricted to, say, the interval $\theta_0 - \frac{1}{2}\pi < \theta < \theta_0 + \frac{1}{2}\pi$, the function $\mathbf{x} = a(\cos \theta) \mathbf{e}_1 + a(\sin \theta) \mathbf{e}_2$ is one-to-one.

Example 3.4.

$$\text{The function} \quad x_1 = t^2, \quad x_2 = \begin{cases} 0, & \text{if } t \leq 0 \\ t^2 \sin 1/t, & \text{if } t > 0 \end{cases} \quad -\infty < t < \infty$$

shown in Fig. 3-3 has continuous derivatives for all t ; however, at $t = 0$, $dx_1/dt = dx_2/dt = 0$ and hence it is not a regular representation. Note that this function has multiple points in *every* neighborhood of $t = 0$. For consider an arbitrary $\delta > 0$. Select an integer $N > 0$ so that $1/2\pi N < \delta$ and consider $t_1 = -(1/2\pi N)$ and $t_2 = +(1/2\pi N)$. Clearly, $-\delta < t_1 < t_2 < \delta$; moreover,

$$x_1(t_1) = 1/4\pi^2 N^2 = x_1(t_2)$$

$$\text{and} \quad x_2(t_1) = 0 = (1/4\pi^2 N^2) \sin 2\pi N = x_2(t_2)$$

Thus we have a multiple point in $-\delta < t < \delta$.

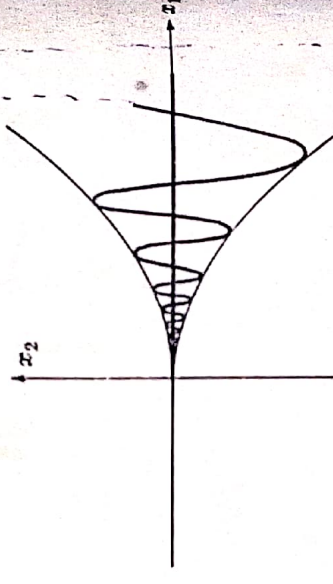


Fig. 3-3

REGULAR CURVES

A real valued function $t = t(\theta)$ on an interval I_θ is an *allowable change of parameter* if

- (i) $t(\theta)$ is of class C^1 in I_θ ,
- (ii) $dt/d\theta \neq 0$ for all θ in I_θ .

Note that if $t = t(\theta)$ is an allowable change of parameter on I_θ , then $dt/d\theta$ is continuous and $dt/d\theta \neq 0$. Hence either $dt/d\theta > 0$ on I_θ , in which case $t(\theta)$ is a smooth increasing function, or $dt/d\theta < 0$ on I_θ and $t(\theta)$ is a smooth decreasing function. This will enable us to prove (Problem 3.13)

Theorem 3.2. If $t = t(\theta)$ is an allowable change of parameter on I_θ , then

- (i) $t = t(\theta)$ is a one-to-one mapping of I_θ onto an interval $I_t = t(I_\theta)$.
- (ii) The inverse function $\theta = \theta(t)$ is an allowable change of parameter on I_t .

Example 3.5.

- (a) The function $t = (b - a)\theta + a$, $0 \leq \theta \leq 1$, $a < b$, is an allowable change of parameter which takes the interval $0 \leq \theta \leq 1$ onto $a \leq t \leq b$. The inverse $\theta = (t - a)/(b - a)$ is an allowable change of parameter which takes $a \leq t \leq b$ onto $0 \leq \theta \leq 1$.
- (b) The function $t = \tan(\pi\theta/2)$, $0 \leq \theta < 1$, is an allowable change of parameter which takes $0 \leq \theta < 1$ onto $0 \leq t < \infty$. The inverse is $\theta = (2/\pi) \tan^{-1} t$ which takes $0 \leq t < \infty$ onto $0 \leq \theta < 1$.

A regular parametric representation $\mathbf{x} = \mathbf{x}(t)$, $t \in I$, is *equivalent* to a regular parametric representation $\mathbf{x} = \mathbf{x}^*(\theta)$, $\theta \in I_\theta$, if there exists an allowable change of parameter $t = t(\theta)$ on I_θ such that

$$(i) \quad t(I_\theta) = I, \quad (ii) \quad \mathbf{x}(t(\theta)) = \mathbf{x}^*(\theta)$$

In Problem 3.14 we show that this defines an equivalence relation on the set of regular representations. We define a *regular curve to be an equivalence class of regular parametric representations*.

Note that a representation $\mathbf{x} = \mathbf{x}(t)$ uniquely determines a curve C consisting of all representations related to it by an allowable change of parameter. Thus we may say "the curve C given by $\mathbf{x} = \mathbf{x}(t) \dots$ ". However, a property of $\mathbf{x} = \mathbf{x}(t)$ may not necessarily be a property of the curve. Any property of the curve must be common to all representations or, as we say, "independent of the parameter."

Example 3.6.

Suppose we introduce the allowable change in parameter $\theta = t + 1$, $-1 \leq t \leq 2\pi - 1$, in the representation

$$\mathbf{x} = (\cos \theta)(2 \cos \theta - 1)\mathbf{e}_1 + (\sin \theta)(2 \cos \theta - 1)\mathbf{e}_2, \quad 0 \leq \theta \leq 2\pi$$

of Example 3.2. This gives the equivalent parametric representation

$$\mathbf{x} = [\cos(t+1)][2 \cos(t+1) - 1]\mathbf{e}_1 + [\sin(t+1)][2 \cos(t+1) - 1]\mathbf{e}_2, \quad -1 \leq t \leq 2\pi - 1$$

As t increases through the interval $-1 \leq t \leq 2\pi - 1$, the quantity $\theta = t + 1$ increases smoothly through $0 \leq \theta \leq 2\pi$, and the transformed equation traces the same set of points in the same direction as before, as shown in Fig. 3-4(a). If we introduce the allowable change in parameter $\theta = -t$, $-2\pi \leq t \leq 0$, we obtain the equivalent representation

$$\mathbf{x} = (\cos t)(2 \cos t - 1)\mathbf{e}_1 - (\sin t)(2 \cos t - 1)\mathbf{e}_2, \quad -2\pi \leq t \leq 0$$

Here as t increases through $-2\pi \leq t \leq 0$, the quantity $\theta = -t$ decreases through the interval $0 \leq \theta \leq 2\pi$ and the set of points is traced in the opposite direction or *sense* as shown in Fig. 3-4(b). Thus the sense in which a curve is traced is a property of the representation and not of the curve. If we introduce the change of parameter

$$\theta = \theta(t) = \begin{cases} t, & \text{for } 0 \leq t \leq \pi/3 \\ -t + 2\pi, & \text{for } \pi/3 < t < 5\pi/3 \\ t, & \text{for } 5\pi/3 \leq t \leq 2\pi \end{cases}$$

which we note is not an allowable change in parameter, the transformed equation

$$\mathbf{x} = [\cos \theta(t)][2 \cos \theta(t) - 1]\mathbf{e}_1 + [\sin \theta(t)][2 \cos \theta(t) - 1]\mathbf{e}_2, \quad 0 \leq t \leq 2\pi$$

will trace the same set of points but in the direction shown in Fig. 3-4(c). With our definition this curve is *not* the same as the one above. Thus a curve should be thought of not simply as a set of points in E^3 but as a general method of traversing the set of points specified by a collection of equivalent parametric representations.

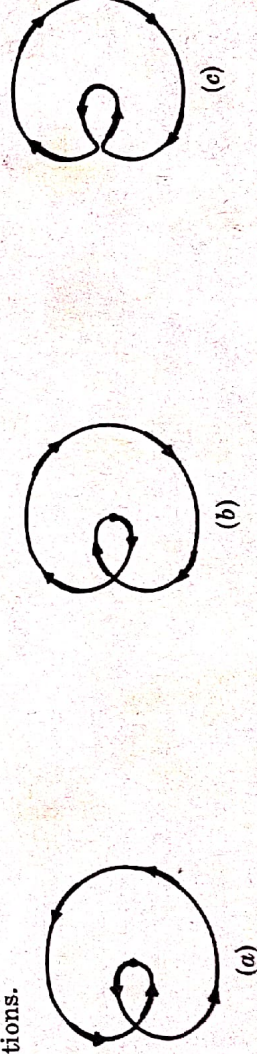


Fig. 3-4

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Example 3.7.

An important example of a space curve is the circular helix

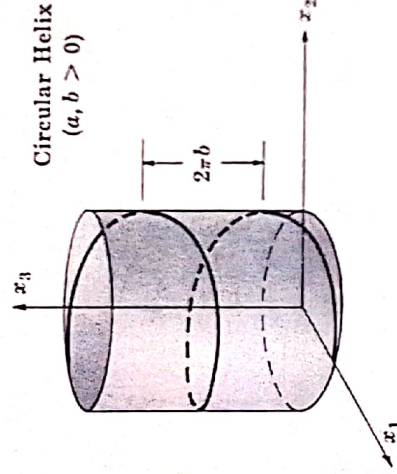
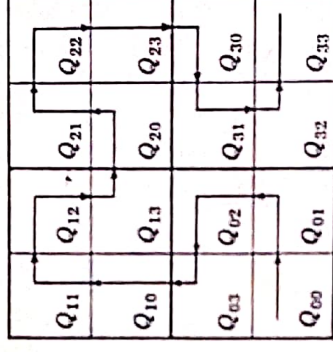
$$\mathbf{x} = a(\cos t)\mathbf{e}_1 + a(\sin t)\mathbf{e}_2 + b t\mathbf{e}_3, \quad a, b \neq 0, \quad -\infty < t < \infty$$

or $x_1 = a \cos t, \quad x_2 = a \sin t, \quad x_3 = bt, \quad a, b \neq 0, \quad -\infty < t < \infty$

shown in Fig. 3-5. The curve lies on the right circular cylinder of radius $|a|$: $x_1 = a \cos t, x_2 = a \sin t, -\infty < x_3 < \infty$. The equation $x_3 = bt$ "moves" the points of the curve uniformly in the x_3 direction. When t increases by 2π , then x_1 and x_2 return to their original values, while x_3 increases ($b > 0$) or decreases ($b < 0$) by $2\pi|b|$, the *pitch* of the helix.

Example 3.8.

It is interesting to note that the unit interval $0 \leq t \leq 1$ can be mapped *continuously* onto the unit square Q : $0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1$ in E^2 , giving a "curve" which fills a two dimensional region. Such a mapping, called a Peano curve, is constructed as follows. We divide Q into four equal squares which together with their boundaries are denoted by Q_0, Q_1, Q_2, Q_3 . Suppose further that each of the Q_i is again subdivided into four equal squares $Q_{i0}, Q_{i1}, Q_{i2}, Q_{i3}$, and each of the latter subdivided again, etc. We also suppose the squares are indexed so that if we pass through the squares in the order of increasing subscripts, we obtain an arc which does not cross itself, as shown in Fig. 3-6. That this can be done is left to the reader as an exercise.

**Fig. 3-5****Fig. 3-6**

Every t_0 in $0 \leq t \leq 1$ can be expressed uniquely as an infinite decimal:

$$t_0 = .a_1 a_2 a_3 \dots = \frac{a_1}{10} + \frac{a_2}{10^2} + \frac{a_3}{10^3} + \dots$$

The same is true with base four, that is, each t_0 can be expressed uniquely as a series

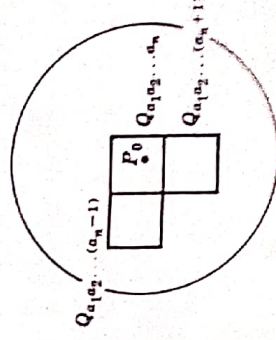
$$t_0 = \frac{a_1}{4} + \frac{a_2}{4^2} + \frac{a_3}{4^3} + \dots$$

with integers $0 \leq a_i \leq 3$.

(Uniqueness is obtained by omitting series ending in 3's, i.e. $\frac{1}{4} + \frac{1}{4^2} + \frac{3}{4^3} + \frac{3}{4^4} + \dots = \frac{1}{4} + \frac{2}{4^2} + \frac{2}{4^3} + \frac{2}{4^4} + \dots$)

To each $t_0 = \sum \frac{a_i}{4^i}$ we now assign the unique point P_0 in Q which is common to the infinite sequence of closed nested squares $Q_{a_1}, Q_{a_1 a_2}, Q_{a_1 a_2 a_3}, \dots$. This mapping is onto Q , for it can be shown that every P in Q will be a point common to an allowable sequence of nested intervals.

Finally, this mapping is continuous. For let $S_\epsilon(P_0)$ be an arbitrary neighborhood of P_0 . As shown in Fig. 3-7, we choose $Q_{a_1 a_2 \dots a_n}$ containing P_0 so small that it and its adjacent squares of the same size

**Fig. 3-7**

are contained in $S_\epsilon(P_0)$. But then all t in the following open interval containing t_0

$$\frac{a_1}{4} + \frac{a_2}{4^2} + \cdots + \frac{a_n - 1}{4^n} < t < \frac{a_1}{4} + \frac{a_2}{4^2} + \cdots + \frac{a_n + 1}{4^n}$$

will be mapped into $S_\epsilon(P_0)$. Hence the mapping is continuous.

This mapping is not one-to-one since points on the boundaries of the squares are common to more than one sequence of nested squares. In fact it can be shown that no continuous mapping of the line onto the square can be one-to-one.

A regular curve $\mathbf{x} = \mathbf{x}(t)$, $t \in I$, is said to be *simple* if there are no multiple points; that is, if $t_1 \neq t_2$ implies $\mathbf{x}(t_1) \neq \mathbf{x}(t_2)$. This is clearly a property of the curve, not of the representation.

A regular curve $\mathbf{x} = \mathbf{x}(t)$, $t \in I$, is called a *regular arc* if I is a closed interval $a \leq t \leq b$. The points $\mathbf{x}(a)$ and $\mathbf{x}(b)$ are called the *end points* of the arc. An *arc segment* of a curve $\mathbf{x} = \mathbf{x}(t)$ on I is an arc $\mathbf{x} = \mathbf{x}^*(t)$, $t \in I^*$, where I^* is any closed interval contained in I and $\mathbf{x}^*(t)$ is the restriction of $\mathbf{x}(t)$ to I^* .

Example 3.9.

The curve in Example 3.2 is a regular arc, since

$$\begin{aligned} x_1 &= (\cos \theta)(2 \cos \theta - 1) & 0 \leq \theta \leq 2\pi \\ x_2 &= (\sin \theta)(2 \cos \theta - 1) \end{aligned}$$

is a regular representation on the closed interval $0 \leq \theta \leq 2\pi$. Observe that here the end points of the curve are equal. The part of the curve restricted to, say, the interval $0 \leq \theta \leq \pi$ is a simple arc segment of the curve and is shown in Fig. 3-8.

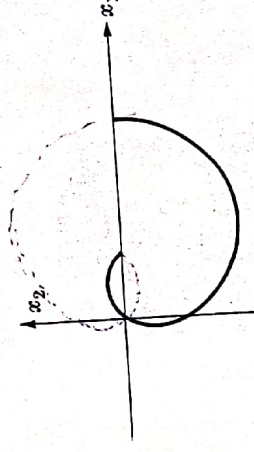


Fig. 3-8

Finally, let $\mathbf{x} = \mathbf{x}(t)$ and $\mathbf{x} = \mathbf{x}^*(\theta)$ be two representations of a regular curve. If $dt/d\theta > 0$, then t increases with increasing θ , and $\mathbf{x} = \mathbf{x}(t)$ and $\mathbf{x} = \mathbf{x}^*(\theta)$ trace the curve in the same direction. If $dt/d\theta < 0$, then t decreases with increasing θ , and $\mathbf{x} = \mathbf{x}(t)$ and $\mathbf{x} = \mathbf{x}^*(\theta)$ will trace the curve in opposite directions. A *regular oriented curve* is a collection along which a specific direction is chosen. That is, a regular oriented curve is a collection of regular parametric representations any two of which are related by an allowable change of parameter having a positive derivative.

ORTHOGONAL PROJECTIONS

Let $\mathbf{x} = \mathbf{x}(t)$ represent a curve C , as shown in

Fig. 3-9. For a fixed t_0 the equation

$$-\infty < k < \infty$$

$$\mathbf{x} = x_1(t_0)\mathbf{e}_1 + x_2(t_0)\mathbf{e}_2 + k\mathbf{e}_3,$$

or

$$-\infty < k < \infty$$

$$x_1 = x_1(t_0), \quad x_2 = x_2(t_0), \quad x_3 = k,$$

is the equation of the line orthogonal to the x_1x_2 plane and passing through the point $\mathbf{x}(t_0)$. It follows that the family of lines

$$-\infty < k < \infty \quad (3.3)$$

$$x_1 = x_1(t), \quad x_2 = x_2(t), \quad x_3 = k,$$

generate a cylindrical surface orthogonal to the x_1x_2 plane and containing the curve C .

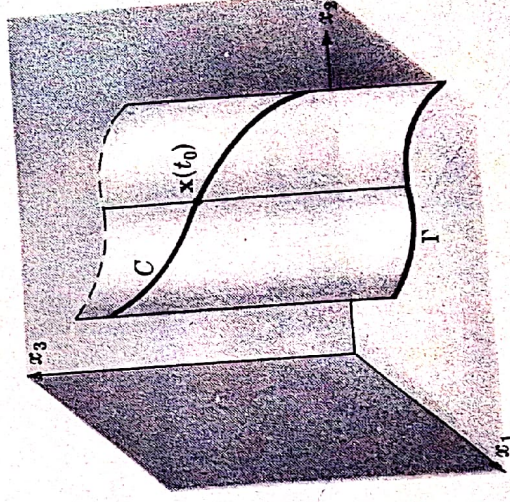


Fig. 3-9

The intersection of the cylinder (3.3) with the x_1x_2 plane, $x_3 = 0$, is the *orthogonal projection* Γ of $\mathbf{x} = \mathbf{x}(t)$ onto the x_1x_2 plane. Hence the orthogonal projection of $\mathbf{x} = \mathbf{x}(t)$ is given by

$$x_1 = x_1(t), \quad x_2 = x_2(t), \quad x_3 = 0$$

The orthogonal projections of $\mathbf{x} = \mathbf{x}(t)$ onto the x_2x_3 and x_1x_3 planes are respectively

$$x_1 = 0, \quad x_2 = x_2(t), \quad x_3 = x_3(t)$$

$$x_1 = x_1(t), \quad x_2 = 0, \quad x_3 = x_3(t)$$

and

Example 3.10.

The orthogonal projection of the space curve $x_1 = t$, $x_2 = t^2$, $x_3 = t^3$, $-\infty < t < \infty$, onto the x_1x_2 plane is the parabola $x_1 = t$, $x_2 = t^2$, $x_3 = 0$. The projection onto the x_1x_3 plane is the cubic $x_1 = t$, $x_3 = t^3$, $x_2 = 0$. The curve itself is the intersection of the two cylinders

$$x_1 = t, \quad x_2 = t^2, \quad -\infty < x_3 < \infty \quad \text{and} \quad x_1 = t, \quad x_3 = t^3, \quad -\infty < x_2 < \infty$$

as shown in Fig. 3-10.

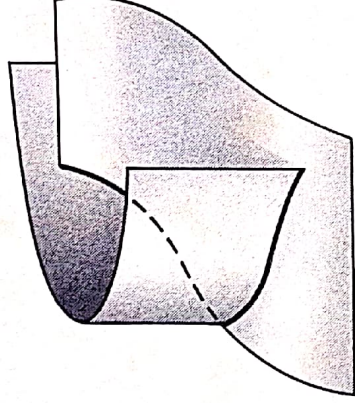


Fig. 3-10

IMPLICIT REPRESENTATIONS OF CURVES

A curve in space can be determined as the intersection of two surfaces, i.e. as those points (x_1, x_2, x_3) satisfying two relations of the form

$$F_1(x_1, x_2, x_3) = 0 \quad \text{and} \quad F_2(x_1, x_2, x_3) = 0 \quad (3.4)$$

If at a point (x_1, x_2, x_3) satisfying the above,

$$\det \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} \end{pmatrix} \neq 0$$

then it follows from the implicit function theorem that for some neighborhood of x_3 we can solve (3.4) for x_1 and x_2 as functions of x_3 , obtaining a representation of the form

$$x_1 = x_1(x_3), \quad x_2 = x_2(x_3), \quad x_3 = x_3$$

with x_3 itself the parameter. This defines at least locally a regular curve.

Example 3.11.

The intersection of the two second degree surfaces $x_2 - x_3^2 = 0$ and $x_3x_1 - x_2^2 = 0$ is the third degree curve $x_1 = t^3$, $x_2 = t^2$, $x_3 = t$ together with the x_1 axis, $x_1 = t$, $x_2 = 0$, $x_3 = 0$. These are obtained as follows. For $x_3 \neq 0$ we can solve the given relations for x_1 and x_2 in terms of x_3 , obtaining

$$x_2 = x_3^2, \quad x_1 = x_2^2/x_3 = x_3^4/x_3 = x_3^3$$

or, if we let $x_3 = t$,

$$x_1 = t^3, \quad x_2 = t^2, \quad x_3 = t$$

If $x_3 = 0$, then $x_2 = x_3^2 = 0$ and x_1 can be arbitrary. This gives the x_1 axis, $x_1 = t$, $x_2 = 0$, $x_3 = 0$. Observe that the point $(0, 0, 0)$ is the intersection of the two curves.

REGULAR CURVES OF CLASS C^m

We define a regular parametric representation $\mathbf{x} = \mathbf{x}(t)$ on I to be a *regular parametric representation of class C^m* ($m \geq 1$) if $\mathbf{x}(t)$ is of class C^m on I . Similarly, an allowable change of parameter $t = t(\theta)$ on I_θ shall be called an *allowable change of parameter of class C^m* if $t(\theta)$ is of class C^m on I_θ . Finally, two regular representations of class C^m define the same *regular curve of class C^m* if they are related by an allowable change of parameter of class C^m . Thus a regular curve of class C^m is a collection of representations of class C^m any two of which are related by an allowable change of parameter of class C^m .

Although a representation $\mathbf{x} = \mathbf{x}(t)$ of class C^m is also of class C^j for all $j \leq m$, the *curve $\mathbf{x} = \mathbf{x}(t)$* of class C^m is not a curve of class C^j for $j < m$, since the curve $\mathbf{x} = \mathbf{x}(t)$ of class C^m contains only representations related to $\mathbf{x} = \mathbf{x}(t)$ by allowable changes of parameter of class C^m , whereas the curve $\mathbf{x} = \mathbf{x}(t)$ of class C^j for $j < m$ contains in addition representations related to $\mathbf{x} = \mathbf{x}(t)$ by allowable changes of parameter which are, say, of class C^j but not of class C^m .

Example 3.12.

The vector function $\mathbf{w}(t) = a(\cos t)\mathbf{e}_1 + a(\sin t)\mathbf{e}_2 + bte_3$, $-\infty < t < \infty$, is analytic. Thus the helix $\mathbf{x} = \mathbf{w}(t)$ can be considered as a regular analytic curve provided we consider only those representations which are related to it by an analytic change in parameter.

Example 3.13.

The representation

$$\mathbf{x} = \begin{cases} te_1 + e^{-1/t^2} e_3 & \text{for } t < 0 \\ 0 & \text{for } t = 0 \\ te_1 + e^{-1/t^2} e_2 & \text{for } t > 0 \end{cases}$$

is of class C^∞ (see Example 2.30, page 31), and together with all related representations of class C^∞ defines the curve of class C^∞ shown in Fig. 3-11. Observe that for all $t < 0$ the curve lies in the x_1x_3 plane, and for all $t > 0$ the curve lies in the x_1x_2 plane.

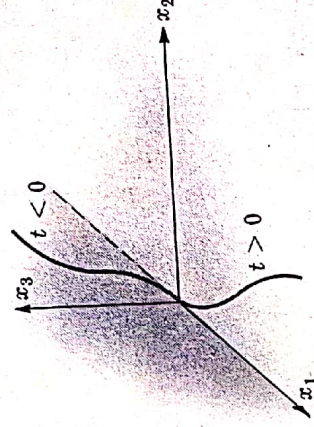


Fig. 3-11

DEFINITION OF ARC LENGTH

The length of an arc is defined in terms of the lengths of approximating polygonal arcs. Namely, let an arc C , not necessarily regular, be given by $\mathbf{x} = \mathbf{x}(t)$, $a \leq t \leq b$, and consider a subdivision

$$a = t_0 < t_1 < \dots < t_n = b$$

of the interval $a \leq t \leq b$. This determines a sequence of points in E^3

$$\mathbf{x}_0 = \mathbf{x}(t_0), \quad \mathbf{x}_1 = \mathbf{x}(t_1), \quad \dots, \quad \mathbf{x}_n = \mathbf{x}(t_n)$$

which are joined in sequence to form an *approximating polygonal arc P* as shown in Fig. 3-12. The length of the line between two adjacent points $\mathbf{x}_{i-1}$ and $\mathbf{x}_i$ is $|\mathbf{x}_i - \mathbf{x}_{i-1}|$. Hence the length of P is

$$s(P) = \sum_{i=1}^n |\mathbf{x}_i - \mathbf{x}_{i-1}| = \sum_{i=1}^n |\mathbf{x}(t_i) - \mathbf{x}(t_{i-1})| \quad (3.5)$$

Now suppose we introduce a better approximating polygonal arc P' by introducing additional points as shown. Since the length of one side of the polygon is less than or equal to the sum of the lengths of the other sides, it follows that the length of P is less than or equal to the length of P' , i.e. $s(P) \leq s(P')$.



Fig. 3-12

Thus we are led to define the length of the arc C as the greatest of lengths of all possible approximating polygonal arcs P . Specifically, an arc $\mathbf{x} = \mathbf{x}(t)$, $a \leq t \leq b$, is said to be *rectifiable* if the set S of all possible $s(P)$ is bounded from above. In this case the s has a *supremum* which is defined to be the *length* of the arc.

A set of S of real numbers is said to be bounded from above if there is a real number M such that $x \leq M$ for all x in S . In this case the number M is called an upper bound of S . Note that if M is an upper bound of S , then any L such that $M \leq L$ is also an upper bound. One of the basic properties of the real numbers is that if S has an upper bound then it has a *least upper bound* or *supremum*, that is, an upper bound s such that if L is an upper bound, then $L \geq s$.

Note that the length of an arc C is independent of the parameter. For let $\mathbf{x} = \mathbf{x}(t)$ on I_t and $\mathbf{x} = \mathbf{x}^*(\theta)$ on I_θ be two representations of C such that $t = t(\theta)$ is one-to-one. To each subdivision $\theta_0 < \theta_1 < \dots < \theta_n$ of I_θ there corresponds a unique subdivision $t_0 < t_1 < \dots < t_n$ of I_t , or, depending upon orientation, $t_n < t_{n-1} < \dots < t_0$, where $t_i = t_i(\theta_i)$, $i = 1, \dots, n$, which gives the same polygonal arc P , and conversely. Thus the set S of the lengths of approximating polygonal arcs is independent of the parameter and hence so is the supremum of S , which is the length of C .

Example 3.14.

The arc $\mathbf{x} = t\mathbf{e}_1 + t^2\mathbf{e}_2$, $0 \leq t \leq 1$, is rectifiable. For consider a subdivision $0 = t_0 < t_1 < \dots < t_n = 1$. The length of the approximating polygonal arc is

$$\begin{aligned} s(P) &= \sum_{i=1}^n |(t_i\mathbf{e}_1 + t_i^2\mathbf{e}_2) - (t_{i-1}\mathbf{e}_1 + t_{i-1}^2\mathbf{e}_2)| \\ &= \sum_{i=1}^n |(t_i - t_{i-1})\mathbf{e}_1 + (t_i^2 - t_{i-1}^2)\mathbf{e}_2| \\ &\leq \sum_{i=1}^n [|t_i - t_{i-1}||\mathbf{e}_1| + |t_i^2 - t_{i-1}^2||\mathbf{e}_2|] \\ &\leq \sum_{i=1}^n [(t_i - t_{i-1}) + (t_i - t_{i-1})(t_i + t_{i-1})] \\ &= \sum_{i=1}^n (t_i - t_{i-1})(1 + t_i + t_{i-1}) \\ &\leq 3 \sum_{i=1}^n (t_i - t_{i-1}) = 3 \end{aligned}$$

where we used the fact that for $0 \leq t_{i-1} < t_i \leq 1$, $1 + t_{i-1} + t_i \leq 3$, and $\sum_{i=1}^n (t_i - t_{i-1}) = t_n - t_0 = 1$.

Thus for any P the quantity $s(P)$ is bounded by 3. Hence the arc is rectifiable and has a length to the supremum of the $s(P)$.

Example 3.15.

The curve

$$\begin{aligned} x_1 &= t \\ x_2 &= \begin{cases} t \cos(1/t) & \text{for } 0 < t \leq 1 \\ 0 & \text{for } t = 0 \end{cases} \quad (0 \leq t \leq 1) \end{aligned}$$

shown in Fig. 3-13 below is not rectifiable. For, using the subdivision $0, 1/(N-1)\pi, \dots, 1/2\pi$, we have

$$\begin{aligned} s(P) &= \left| \frac{1}{(N-1)\pi} \mathbf{e}_1 + \frac{1}{(N-1)\pi} [\cos(N-1)\pi] \mathbf{e}_2 \right| \\ &\quad + \left| \left[\frac{1}{(N-2)\pi} - \frac{1}{(N-1)\pi} \right] \mathbf{e}_1 + \left[\frac{1}{(N-2)\pi} \cos(N-2)\pi - \frac{1}{(N-1)\pi} \cos(N-1)\pi \right] \right| \\ &\quad + \dots + \left| \left[1 - \frac{1}{\pi} \right] \mathbf{e}_1 + \left[\cos 1 - \frac{1}{\pi} \cos \pi \right] \mathbf{e}_2 \right| \end{aligned}$$

If we drop the first and last terms, then

$$\begin{aligned}
s(P) &\approx \sum_{n=1}^{N-2} \left| \left[\frac{1}{n\pi} - \frac{1}{(n+1)\pi} \right] \mathbf{e}_1 + \left[\frac{1}{n\pi} \cos n\pi - \frac{1}{n+1} \cos (n+1)\pi \right] \mathbf{e}_2 \right| \\
&\approx \sum_{n=1}^{N-2} \left| \left[\frac{1}{n\pi} \cos n\pi - \frac{1}{n+1} \cos (n+1)\pi \right] \mathbf{e}_2 \right| \\
&= \sum_{n=1}^{N-2} \left| (-1)^n \frac{1}{n\pi} - (-1)^{n+1} \frac{1}{(n+1)\pi} \right| \\
&= \sum_{n=1}^{N-2} \left| \frac{1}{n\pi} + \frac{1}{(n+1)\pi} \right| \approx \frac{2}{\pi} \sum_{n=1}^{N-2} \frac{1}{n+1}
\end{aligned}$$

where in going from the first to the second line we have used the inequality $|a\mathbf{e}_1 + b\mathbf{e}_2| \geq |b\mathbf{e}_2|$. But the sum $\sum_{n=1}^{N-2} \frac{1}{n+1}$ diverges to infinity. That is, $s(P)$ can be made arbitrarily large by making N sufficiently large. Thus the curve is not rectifiable.

In Problem 3.24 we prove

Theorem 3.3. A regular arc $\mathbf{x} = \mathbf{x}(t)$, $a \leq t \leq b$, is rectifiable and its length is given by the integral

$$s = \int_a^b \left| \frac{d\mathbf{x}}{dt} \right| dt = \int_a^b \sqrt{\left(\frac{dx_1}{dt} \right)^2 + \left(\frac{dx_2}{dt} \right)^2 + \left(\frac{dx_3}{dt} \right)^2} dt \quad (3.6)$$

Example 3.16.

The length of the arc segment of the helix $\mathbf{x} = (a \cos t)\mathbf{e}_1 + (a \sin t)\mathbf{e}_2 + bte_3$, $0 \leq t \leq 2\pi$, is

$$s = \int_0^{2\pi} \sqrt{a^2 \sin^2 t + a^2 \cos^2 t + b^2} dt = \int_0^{2\pi} (a^2 + b^2)^{1/2} dt = 2\pi(a^2 + b^2)^{1/2}$$

ARC LENGTH AS A PARAMETER

Let $\mathbf{x} = \mathbf{x}(t)$ be a regular curve on I , and consider the function

$$s = s(t) = \int_{t_0}^t \left| \frac{d\mathbf{x}}{dt} \right| dt \quad (3.7)$$

If $t \geq t_0$, then $s \geq 0$ and is equal to the length of the arc segment of the curve between $\mathbf{x}(t_0)$ and $\mathbf{x}(t)$. If $t < t_0$, then $s < 0$ and is equal to minus the length of the arc segment between $\mathbf{x}(t_0)$ and $\mathbf{x}(t)$.

Now, it follows from the Fundamental Theorem of Calculus that (3.7) has a continuous nonvanishing derivative given by

$$\frac{ds}{dt} = \frac{d}{dt} \int_{t_0}^t \left| \frac{d\mathbf{x}}{dt} \right| dt = \left| \frac{d\mathbf{x}}{dt} \right|$$

Hence $s = s(t)$ is an allowable change of parameter on I . Also $s(t)$ is of class C^m on I if $\mathbf{x}(t)$ is of class C^m . Thus the arc length s can be introduced along the curve as a parameter.

Note that a representation in terms of arc length is not unique, for it depends upon the chosen initial point t_0 (where $s = 0$) and on orientation; that is, we could take

$$s(t) = \int_t^{t_0} \left| \frac{d\mathbf{x}}{dt} \right| dt = - \int_{t_0}^t \left| \frac{d\mathbf{x}}{dt} \right| dt$$

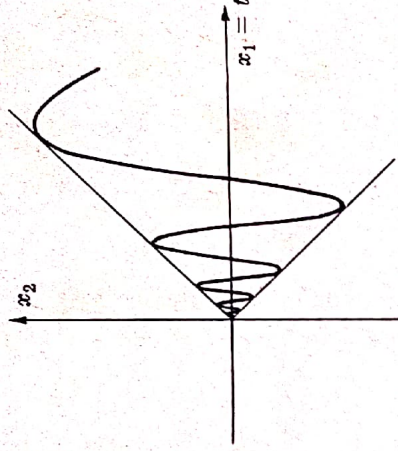


Fig. 3-13

To be precise, therefore, we define a representation $\mathbf{x} = \mathbf{x}(s)$ on I_s to be a *representation in terms of arc length* or a *natural representation* if $|\mathbf{dx}/ds| = 1$. From this we prove (Problems 3.19 and 3.20)

Theorem 3.4. If $\mathbf{x} = \mathbf{x}(s)$ is a natural representation of a curve C , then

- (i) $|s_2 - s_1|$ is the length of the arc segment of C between $\mathbf{x}(s_1)$ and $\mathbf{x}(s_2)$.
- (ii) If $\mathbf{x} = \mathbf{x}^*(s^*)$ is any other natural representation of C , then $s = \pm s^* + \text{constant}$.
- (iii) If $\mathbf{x} = \mathbf{x}^*(t)$ is any representation of C of the same orientation as $\mathbf{x} = \mathbf{x}(s)$, then $ds/dt = |\mathbf{dx}/dt|$. Otherwise $ds/dt = -|\mathbf{dx}/dt|$.

Note finally that if $s = s(t)$ is defined by the integral in (3.7), then $\mathbf{x} = \mathbf{x}(t(s))$ is a natural representation since

$$\left| \frac{d\mathbf{x}}{ds} \right| = \left| \frac{d\mathbf{x}}{dt} \left\| \frac{dt}{ds} \right\| \right| = \left| \frac{d\mathbf{x}}{dt} \right| \left| \frac{ds}{dt} \right| = \left| \frac{d\mathbf{x}}{dt} \right| \left| \frac{dx}{dt} \right| = 1$$

Example 3.17.

To obtain a natural representation of the helix

$$\mathbf{x} = (a \cos t)\mathbf{e}_1 + (a \sin t)\mathbf{e}_2 + bte_3$$

we consider

$$s = \int_0^t \left| \frac{d\mathbf{x}}{dt} \right| dt = \int_0^t (a^2 + b^2)^{1/2} dt = (a^2 + b^2)^{1/2} t$$

If we substitute $t = (a^2 + b^2)^{-1/2}s$ into the above, we obtain the natural representation

$$\mathbf{x} = a \cos [(a^2 + b^2)^{-1/2}s] \mathbf{e}_1 + a \sin [(a^2 + b^2)^{-1/2}s] \mathbf{e}_2 + b(a^2 + b^2)^{-1/2} s \mathbf{e}_3$$

Unless otherwise stated, differentiation with respect to a natural parameter s will be denoted by dots and differentiation with respect to any other parameter will be denoted by primes; for example,

$$\dot{\mathbf{x}} = \frac{d\mathbf{x}}{ds}, \quad \ddot{\mathbf{x}} = \frac{d^2\mathbf{x}}{ds^2}, \quad \mathbf{x}' = \frac{d\mathbf{x}}{dt}, \quad \mathbf{x}'' = \frac{d^2\mathbf{x}}{dt^2}, \text{ etc.}$$

Solved Problems

REGULAR REPRESENTATIONS

3.1. Show that $\mathbf{x} = te_1 + (t^2 + 1)e_2 + (t - 1)^3e_3$ is a regular parametric representation for all t and find the projections onto the x_1x_2 and x_1x_3 planes.

$\mathbf{dx}/dt = \mathbf{e}_1 + 2te_2 + 3(t - 1)^2e_3$ is continuous and $|\mathbf{dx}/dt| = [1 + 4t^2 + 9(t - 1)^4]^{1/2} \neq 0$ for all t . Hence $\mathbf{x}$ is regular for all t . The projection onto the x_1x_2 plane is the parabola $x_1 = t$, $x_2 = t^2 + 1$, $x_3 = 0$, or $x_2 = x_1^2 + 1$, $x_3 = 0$. The projection onto the x_1x_3 plane is the cubic $x_1 = t$, $x_3 = (t - 1)^3$, $x_2 = 0$, or $x_3 = (x_1 - 1)^3$, $x_2 = 0$. The curve is the intersection of the cylinders $x_2 = x_1^2 + 1$ and $x_3 = (x_1 - 1)^3$.

3.2. Show that the representation $x_1 = (1 + \cos \theta)$, $x_2 = \sin \theta$, $x_3 = 2 \sin(\theta/2)$, $-2\pi \leq \theta \leq 2\pi$ is regular and lies on the sphere of radius 2 about the origin and the cylinder $(x_1 - 1)^2 + x_2^2 = 1$.

$\frac{dx_1}{d\theta} = -\sin \theta$, $\frac{dx_2}{d\theta} = \cos \theta$, $\frac{dx_3}{d\theta} = \cos(\theta/2)$
 are continuous and

$$\left[\left(\frac{dx_1}{d\theta} \right)^2 + \left(\frac{dx_2}{d\theta} \right)^2 + \left(\frac{dx_3}{d\theta} \right)^2 \right]^{1/2} = [1 + \cos^2(\theta/2)]^{1/2} \neq 0$$

Hence the representation is regular. Since

$$\begin{aligned} x_1^2 + x_2^2 + x_3^2 &= (1 + \cos \theta)^2 + \sin^2 \theta + 4 \sin^2(\theta/2) \\ &= (1 + \cos \theta)^2 + \sin^2 \theta + 2(1 - \cos \theta) = 4 \end{aligned}$$

and $(x_1 - 1)^2 + x_2^2 = \cos^2 \theta + \sin^2 \theta = 1$, the curve lies on the sphere of radius 2 and the circular cylinder $(x_1 - 1)^2 + x_2^2 = 1$. It is the intersection of these surfaces as shown in Fig. 3-14.

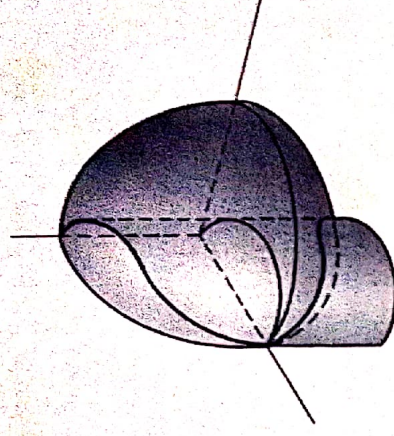


Fig. 3-14

3.3. The equation of the *caisson* of *Diocles* in polar coordinates is $r = 2 \sin \theta \tan \theta$, $-\pi/2 < \theta < \pi/2$. Sketch the curve and find a parametric representation in rectangular coordinates.

θ	$2 \sin \theta$	$\tan \theta$	r
$-\pi/2 +$	-2	$-\infty$	∞
$-\pi/4$	$-\sqrt{2}$	-1	$\sqrt{2}$
0	0	0	0
$\pi/4$	$\sqrt{2}$	1	$\sqrt{2}$
$\pi/2 -$	2	$+\infty$	∞

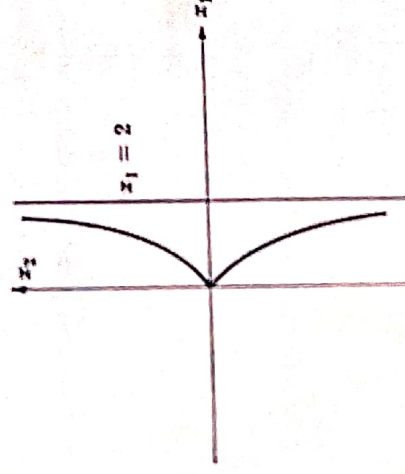


Fig. 3-15

Since $x_1 = r \cos \theta$ and $x_2 = r \sin \theta$, we have the parametric representation

$$x_1 = 2 \sin^2 \theta, \quad x_2 = 2 \sin^2 \theta \tan \theta, \quad -\pi/2 < \theta < \pi/2$$

Observe that $x_1 \rightarrow 2$ as $\theta \rightarrow -\pi/2$ or as $\theta \rightarrow \pi/2$. Note that the representation above is not regular at $\theta = 0$, since both

$$\frac{dx_1}{d\theta} = 4 \sin \theta \cos \theta \quad \text{and} \quad \frac{dx_2}{d\theta} = 2 \sin^2 \theta \sec^2 \theta + 4 \sin \theta \cos \theta \tan \theta$$

vanish at $\theta = 0$.

3.4. The *epicycloid* is a plane curve generated by a point P on the circumference of a circle C as C rolls without slipping on the exterior of a fixed circle C_0 as shown in Fig. 3-16. Find a parametric representation of the epicycloid when C has radius r , C_0 is at the origin with radius r_0 , and P is initially located at $(r_0, 0)$.

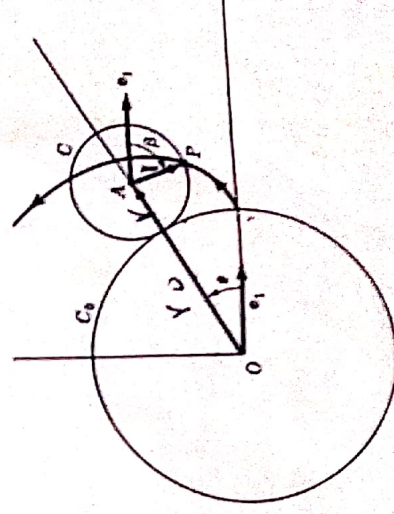


Fig. 3-16

Let A denote the center of C and θ the angle that OA makes with e_1 . Then

$$OA = |OA|(\cos \theta)e_1 + |OA|(\sin \theta)e_2$$

$$= (r_0 + r)(\cos \theta)e_1 + (r_0 + r)(\sin \theta)e_2$$

If β is the angle that $\mathbf{AP}$ makes with $\mathbf{e}_1$, then

$$\beta = \angle OAP + \theta - \pi \quad \text{or} \quad \beta = \frac{r_0}{r} + \theta - \pi = \frac{r_0 + r}{r} \theta - \pi$$

It follows that

$$\begin{aligned} \mathbf{AP} &= |\mathbf{AP}|(\cos \beta)\mathbf{e}_1 + |\mathbf{AP}|(\sin \beta)\mathbf{e}_2 = r \left[\cos \left(\frac{r_0 + r}{r} \theta - \pi \right) \right] \mathbf{e}_1 + r \left[\sin \left(\frac{r_0 + r}{r} \theta - \pi \right) \right] \mathbf{e}_2 \\ &= -r \left[\cos \left(\frac{r_0 + r}{r} \theta \right) \right] \mathbf{e}_1 - r \left[\sin \left(\frac{r_0 + r}{r} \theta \right) \right] \mathbf{e}_2 \end{aligned}$$

Hence

$$\begin{aligned} \mathbf{x} = \mathbf{OP} &= \mathbf{OA} + \mathbf{AP} = \left[(r_0 + r) \cos \theta - r \cos \left(\frac{r_0 + r}{r} \theta \right) \right] \mathbf{e}_1 \\ &\quad + \left[(r_0 + r) \sin \theta - r \sin \left(\frac{r_0 + r}{r} \theta \right) \right] \mathbf{e}_2 \end{aligned}$$

which is the required result.

3.5. If in the preceding problem $r_0 = 3$ and $r = 1$, the equation of the epicycloid is

$$x_1 = 4 \cos \theta - \cos 4\theta, \quad x_2 = 4 \sin \theta - \sin 4\theta$$

Determine the singular (non-regular) points and sketch.

$$dx_1/d\theta = -4 \sin \theta + 4 \sin 4\theta = 0 \quad \text{iff} \quad \sin \theta = \sin 4\theta, \quad \text{or} \quad \theta = 2n\pi/3, (2n+1)\pi/5, n = 0, \pm 1, \dots$$

$$dx_2/d\theta = 4 \cos \theta - 4 \cos 4\theta = 0 \quad \text{iff} \quad \cos \theta = \cos 4\theta, \quad \text{or} \quad \theta = 2n\pi/3, 2n\pi/5, n = 0, \pm 1, \dots$$

It follows that both derivatives vanish if and only if $\theta = 2n\pi/3$, $n = 0, \pm 1, \dots$. Observe that the curve has period 2π .

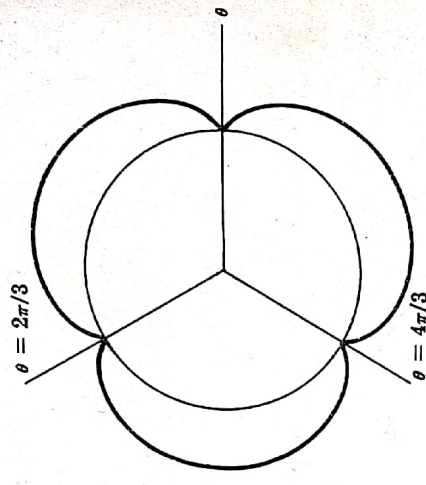


Fig. 3-17

3.6. Determine a parametric representation for the intersection of the cylinder $x_1^2 + x_2^2 = 1$ and the plane $x_1 + x_2 + x_3 = 1$ that does not involve radicals.

We are led to take $x_1 = \cos \theta$ and $x_2 = \sin \theta$, $0 \leq \theta \leq 2\pi$. Then $x_3 = 1 - x_1 - x_2 = 1 - \cos \theta \sin \theta$. Thus

$$\mathbf{x} = \cos \theta \mathbf{e}_1 + \sin \theta \mathbf{e}_2 + (1 - \cos \theta - \sin \theta) \mathbf{e}_3, \quad 0 \leq \theta \leq 2\pi$$

is a parametric representation of the intersection.

3.7. If $g(t)$ is continuous at $t = t_0$ and $g(t_0) \neq 0$, show there exists a $\delta > 0$ such that $g(t) \neq 0$ for t in $S_\delta(t_0)$. Use this to prove Theorem 3.1: If $\mathbf{x} = \mathbf{x}(t)$ is a regular representation on I , then for every t_0 in I there exists a neighborhood of t_0 in which $\mathbf{x}(t)$ is one-to-one.

Take $\epsilon = \frac{1}{2}|g(t_0)|$. Since $g(t)$ is continuous at t_0 , there exists a $\delta > 0$ such that $|g(t) - g(t_0)| < \epsilon$ for t in $S_\delta(t_0)$. Hence for t in $S_\delta(t_0)$,

$$|g(t_0)| = |g(t_0) - g(t) + g(t)| \leq |g(t_0) - g(t)| + |g(t)| \leq \epsilon + |g(t)| \leq \frac{1}{2}|g(t_0)| + |g(t)|$$

or $|g(t)| \geq \frac{1}{2}|g(t_0)|$. Since $g(t_0) \neq 0$, then $g(t) \neq 0$ for t in $S_\delta(t_0)$.

Since $\mathbf{x} = \mathbf{x}(t)$ is regular on I and t_0 is in I , at least one of the derivatives, say $x'_1(t_0) \neq 0$. Also $x'_1(t)$ is continuous at t_0 . It follows that there exists a $\delta > 0$ such that $x'_1(t) \neq 0$ for all t in $S_\delta(t_0)$. Now, in $S_\delta(t_0)$, $\mathbf{x}(t)$ is one-to-one, for otherwise there would exist $t_1 \neq t_2$ in $S_\delta(t_0)$ such that $\mathbf{x}(t_1) = \mathbf{x}(t_2)$. Hence $x_1(t_1) = x_1(t_2)$. But applying the mean value theorem,

$$0 = \frac{x_1(t_1) - x_1(t_2)}{t_1 - t_2} = x'_1(t'), \quad t_1 < t' < t_2$$

If $\mathbf{x} = \mathbf{x}(t)$ is a regular representation on I with $\dot{\mathbf{x}}(t_0) \neq 0$, show that there exists a neighborhood of t_0 in which $\mathbf{x} = \mathbf{x}(t)$ can be represented implicitly in the form $x_2 = F_1(x_1)$, $x_3 = F_2(x_1)$.

Since $x_1'(t_0) \neq 0$, in some $S_0(t_0)$, $x_1 = x_1(t)$ is one-to-one and has an inverse $t = t(x_1)$. Substitute into the parametric equations $x_1 = x_1(t)$, $x_2 = x_2(t)$, $x_3 = x_3(t)$ to obtain $x_2 = x_2(t(x_1))$, $x_3 = x_3(t(x_1))$ or $x_2 = F_1(x_1)$, $x_3 = F_2(x_1)$.

REGULAR CURVES

3.9. Show that $t = \theta^2/(\theta^2 + 1)$ is an allowable change of parameter on $0 < \theta < \infty$ and takes the interval $0 < \theta < \infty$ onto $0 < t < 1$.

$dt/d\theta = 2\theta/(\theta^2 + 1)^2$ is continuous and $dt/d\theta \neq 0$ on $0 < \theta < \infty$. Hence it is an allowable change of parameter on $0 < \theta < \infty$. Since $\theta^2/(\theta^2 + 1)|_{\theta=0} = 0$ and $\lim_{\theta \rightarrow \infty} \theta^2/(\theta^2 + 1) = 1$, it takes the interval $0 < \theta < \infty$ onto $0 < t < 1$.

3.10. Introduce $t = \tan^{-1}(\theta/4)$ as a parameter along the circle $x_1 = a \cos \theta$, $x_2 = a \sin \theta$,
 $-\pi \leq \theta \leq \pi$.

The half angle identities give

$$\begin{aligned}\cos \theta &= \cos^4(\theta/4) - 6 \cos^2(\theta/4) \sin^2(\theta/4) + \sin^4(\theta/4) \\ &= \frac{1}{(t^2+1)^2} - 6 \frac{t^2}{(t^2+1)^2} + \frac{t^4}{(t^2+1)^2} = \frac{t^4 - 6t^2 + 1}{(t^2+1)^2}\end{aligned}$$

and

$$= \frac{4t(1-t^2)}{(t^2+1)^2} = \frac{t^3}{t^2+1} - 4 \frac{t}{(t^2+1)^2}$$

Thus $x_1 = a \frac{(t^4 - 6t^2 + 1)}{(t^2 + 1)^2}$, $x_2 = \frac{4at(1-t^2)}{(t^2 + 1)^2}$, $-1 \leq t \leq 1$, is the desired representation.

4 — 2 sin θ along the cissoid

parameter $t = 2 \sin \theta$ along the circle $-\pi/2 < \theta < \pi/2$, $x_1 = 2 \sin^2 \theta$, $x_2 = 2 \sin^2 \theta \tan \theta$, the power series expansion of x_1 and x_2

and obtain the first two nonzero terms of the Taylor series for $f(t)$ at the singular point $t = 0$. (See Problem 3.3.)

angular point $t = 0$. (See Problem 5.5.) Observe

$$x_1 = 2(\tfrac{1}{2}t)^2 = \tfrac{1}{2}t^2, \quad x_2 = \tfrac{1}{2}t^2, \quad \text{Thus } x_1 = \tfrac{1}{2}t^2, \quad x_2 = \tfrac{1}{4}t^3 + \tfrac{1}{32}t^5 + o(t^5).$$

on of $(4 - t^2)^{-1/2}$ is $\tfrac{1}{2} + \tfrac{1}{16}t^2 + o(t^2)$. Thus $x_1 = \tfrac{1}{2}t^2$, $x_2 = \tfrac{1}{\sqrt{2}}x_1^{3/2} + o(x_1^2)$.

The expansion of $(4 - t^2)^{-1/2}$ is $\frac{1}{2} + \frac{1}{16}t^2 + o(t^2)$. Thus $x_1 = \frac{1}{2} + \frac{1}{16}t^2 + o(t^2)$. The curve is shaped like the cusp $x_2 = \frac{1}{\sqrt{2}}x_1^{3/2} + o(x_1^{3/2})$.

that in the neighborhood of $\lambda = 0$

that in the case

3.12. Show that there exists an allowable change of parameter $t = t(\theta)$ which takes any interval I onto one of the following three intervals: (i) $0 \leq t \leq 1$, (ii) $0 < t < 1$, (iii) $0 \leq t < 1$. Hence every regular curve has a representation defined in one of the above three intervals.

(iii) $0 \leq t < 1$. Hence every regular interval contains above three intervals.

As shown in Example 3.5, page 45, the linear function $t = (\theta - a)/(b - a)$ is an allowable change of parameter which takes $a \leq \theta \leq b$ onto $0 \leq t \leq 1$, $a < \theta < b$ onto $0 < t < 1$, and $a \leq \theta < b$ onto $0 \leq t < 1$. The linear function $t = -(\theta - a)/(b - a) + 1$ will take $a < \theta \leq b$ onto $0 \leq t < 1$. The function $\theta = \tan^{-1} s$ takes the interval $-\pi/2 < \theta < \pi/2$ onto $-\infty < s < \infty$ onto $0 < t < 1$. It remains to consider $-\pi/2 < \theta < \pi/2$, and $t = (\theta + \frac{1}{2}\pi)/\pi$ takes $-\infty < s < \infty$ onto $0 < t < 1$. Hence the composite function $t = (\pi/2 + \tan^{-1} s)/\pi$ takes $-\infty < s < \infty$ onto $0 < t < 1$. Hence the composite function $\theta = \tan^{-1} s$ takes the interval $a \leq s < \infty$ onto $\tan^{-1} a \leq \theta < \pi/2$ and $t = (\theta - \tan^{-1} a)/(\pi/2 - \tan^{-1} a)$ takes $\tan^{-1} a \leq \theta < \pi/2$ onto $0 \leq t < 1$. We leave the remaining cases as an exercise to the reader. There are allowable changes of

tion $t = \frac{\pi}{2} - \text{Tall}$ for the reader.

Note that all the above functions are *linear* exercises for the reader.

3.13. Prove Theorem 3.2: If $t = t(\theta)$ is an allowable change of parameter on I_θ , then $t(\theta)$ is one-to-one and its inverse $\theta = \theta(t)$ is an allowable change of parameter on $I_t = t(I_\theta)$, if $dt/d\theta \neq 0$, it follows that $dt/d\theta > 0$ or $dt/d\theta < 0$ on I_θ . Since $dt/d\theta$ is continuous and $dt/d\theta \neq 0$, it follows that $dt/d\theta > 0$ or $dt/d\theta < 0$ on I_θ . Suppose $dt/d\theta > 0$ on I_θ ; then $t(\theta)$ is strictly increasing. For if otherwise, i.e. if $t(\theta_1) \equiv t(\theta_2)$ with $\theta_1 < \theta_2$, then, using the mean value theorem,

$$0 \cong \frac{t(\theta_1) - t(\theta_2)}{\theta_1 - \theta_2} = t'(\theta')$$

which is impossible, since $dt/d\theta > 0$ on I_θ . Since $t(\theta)$ is strictly increasing, it is one-to-one and has an inverse $\theta(t)$. Now, since $t(\theta)$ is increasing and continuous, it follows that its inverse $\theta(t)$ is increasing and continuous. This we leave to the reader as an exercise. But then $\theta(t)$ also has derivative

$$\frac{d\theta}{dt} = \lim_{\Delta t \rightarrow 0} \frac{\Delta\theta}{\Delta t} = 1 / \lim_{\Delta\theta \rightarrow 0} \frac{\Delta t}{\Delta\theta} = 1 / \frac{dt}{d\theta}$$

which is continuous and different from zero since $dt/d\theta$ is continuous and different from zero, which completes the proof.

3.14. We recall that a regular parametric representation $\mathbf{x} = \mathbf{x}(t)$ on I_t is defined to be equivalent to a regular parametric representation $\mathbf{x} = \mathbf{x}^*(\theta)$ on I_θ if there exists an allowable change of parameter $t = t(\theta)$ such that $t(I_\theta) = I_t$ and $\mathbf{x}(t(\theta)) = \mathbf{x}^*(\theta)$. Show that this defines an equivalence relation on the set of regular representations.

$\mathbf{x} = \mathbf{x}(t)$ is equivalent to itself under the identity change of parameter $t = \theta$. If $\mathbf{x} = \mathbf{x}(t)$ is equivalent to $\mathbf{x} = \mathbf{x}^*(\theta)$ under the change of parameter $t = t(\theta)$, then $\mathbf{x} = \mathbf{x}^*(\theta)$ is equivalent to $\mathbf{x} = \mathbf{x}(t)$ under the inverse function $\theta = \theta(t)$, since $\theta(I_t) = I_\theta$ and $\mathbf{x}^*(\theta(t)) = \mathbf{x}(t(\theta(t))) = \mathbf{x}(t)$. Finally, suppose $\mathbf{x} = \mathbf{x}(t)$ is equivalent to $\mathbf{x} = \mathbf{x}^*(\theta)$ under $t = t(\theta)$, and $\mathbf{x} = \mathbf{x}^*(\theta)$ is equivalent to $\mathbf{x} = \mathbf{x}^{**}(\phi)$ under $\theta = \theta(\phi)$. Consider the composite function $t = t(\theta(\phi))$. It follows that $\mathbf{x} = \mathbf{x}(t)$ is equivalent to itself under the identity change of parameter $t = \theta$. Hence $t = t(\theta(\phi))$ is an allowable change of parameter on I_ϕ . Also, $t(\theta(I_\phi)) = t(I_\theta) = I_t$ and $\mathbf{x}(t(\theta(\phi))) = \mathbf{x}^*(\theta(\phi)) = \mathbf{x}^{**}(\phi)$. Thus $\mathbf{x} = \mathbf{x}(t)$ is equivalent to $\mathbf{x} = \mathbf{x}^{**}(\phi)$, which completes the proof.

ARC LENGTH

3.15. Compute the length of the arc $\mathbf{x} = 3(\cosh 2t)\mathbf{e}_1 + 3(\sinh 2t)\mathbf{e}_2 + 6t\mathbf{e}_3$, $0 \leq t \leq \pi$.

$$\begin{aligned} s &= \int_0^\pi \left| \frac{d\mathbf{x}}{dt} \right| dt = \int_0^\pi |6 \sinh 2t \mathbf{e}_1 + 6 \cosh 2t \mathbf{e}_2 + 6\mathbf{e}_3| dt \\ &= \int_0^\pi 6[\sinh^2 2t + \cosh^2 2t + 1]^{1/2} dt \\ &= \int_0^\pi 6[2 \cosh^2 2t]^{1/2} dt = \int_0^\pi 6\sqrt{2} \cosh 2t dt = 3\sqrt{2} \sinh 2\pi \end{aligned}$$

3.16. Find the arc length as a function of θ along the epicycloid

$$x_1 = (r_0 + r) \cos \theta - r \cos \left(\frac{r_0 + r}{r} \theta \right), \quad x_2 = (r_0 + r) \sin \theta - r \sin \left(\frac{r_0 + r}{r} \theta \right)$$

See Problem 3.4.

$$\begin{aligned} s &= \int_0^\theta \left[\left(\frac{dx_1}{d\theta} \right)^2 + \left(\frac{dx_2}{d\theta} \right)^2 \right]^{1/2} d\theta \\ &= \int_0^\theta (r_0 + r) \left[\left(-\sin \theta + \sin \left(\frac{r_0 + r}{r} \theta \right) \right)^2 + \left(\cos \theta - \cos \left(\frac{r_0 + r}{r} \theta \right) \right)^2 \right]^{1/2} d\theta \\ &= (r_0 + r) \int_0^\theta \left[2 - 2 \cos \left(\frac{r_0 + r}{r} \theta \right) \right]^{1/2} d\theta = 2(r_0 + r) \int_0^\theta \sin \left(\frac{r_0 + r}{2r} \theta \right) d\theta \\ &= -4 \frac{(r_0 + r)r}{r_0} \cos \left(\frac{r_0 + r}{2r} \theta \right) \Big|_0^\theta = -4 \frac{(r_0 + r)r}{r_0} [1 - \cos(r_0 \theta / 2r)] \end{aligned}$$

Handwritten notes: $\cos^2 + \sin^2 = 1$, $\sin^2 \theta = 1 - \cos^2 \theta$, $\sin \theta = \cos(\theta - \pi/2)$

3.17. Introduce arc length as a parameter along Γ

$$\mathbf{x} = (e^t \cos t)\mathbf{e}_1 + (e^t \sin t)\mathbf{e}_2 + e^t\mathbf{e}_3, \quad -\infty < t < \infty$$

$$\begin{aligned} s &= \int_0^t \left| \frac{d\mathbf{x}}{dt} \right| dt = \int_0^t |(e^t \cos t - e^t \sin t)\mathbf{e}_1 + (e^t \sin t + e^t \cos t)\mathbf{e}_2 + e^t\mathbf{e}_3| dt \\ &= \int_0^t [e^{2t}(-2 \cos t \sin t + 1) + e^{2t}(2 \cos t \sin t + 1) + e^{2t}]^{1/2} dt = \sqrt{3} \int_0^t e^t dt = \sqrt{3}(e^t - 1) \end{aligned}$$

Solving, $t = \log(s/\sqrt{3} + 1)$, $-\sqrt{3} < s < \infty$. Introducing arc length s as a parameter,

$$\mathbf{x} = (s/\sqrt{3} + 1)(\cos \log(s/\sqrt{3} + 1))\mathbf{e}_1 + \sin \log(s/\sqrt{3} + 1)\mathbf{e}_2 + \mathbf{e}_3$$

3.18. Show that

$$\mathbf{x} = \frac{1}{2}(s + \sqrt{s^2 + 1})\mathbf{e}_1 + \frac{1}{2}(s + \sqrt{s^2 + 1})^{-1}\mathbf{e}_2 + \frac{1}{2}\sqrt{2}(\log(s + \sqrt{s^2 + 1}))\mathbf{e}_3$$

is a natural representation, i.e. $|d\mathbf{x}/ds| = 1$.

Let $u = s + \sqrt{s^2 + 1}$. Then $\mathbf{x} = \frac{1}{2}u\mathbf{e}_1 + \frac{1}{2}u^{-1}\mathbf{e}_2 + \frac{1}{2}\sqrt{2}(\log u)\mathbf{e}_3$ and

$$\frac{d\mathbf{x}}{ds} = \frac{d\mathbf{x}}{du} \frac{du}{ds} = \left(\frac{1}{2}\mathbf{e}_1 - \frac{1}{2}u^{-2}\mathbf{e}_2 + \frac{1}{2}\sqrt{2}u^{-1}\mathbf{e}_3 \right) \left(1 + \frac{s}{\sqrt{s^2 + 1}} \right)$$

$$\begin{aligned} \text{Hence } \left| \frac{d\mathbf{x}}{ds} \right| &= \left| \frac{d\mathbf{x}}{du} \right| \left| \frac{du}{ds} \right| = \frac{1}{2}(1 + u^{-4} + 2u^{-2})^{1/2} \frac{s + \sqrt{s^2 + 1}}{\sqrt{s^2 + 1}} \\ &= \frac{1}{2}(1 + u^{-2}) \frac{u}{\sqrt{s^2 + 1}} = \frac{1}{2} \frac{u^2 + 1}{u\sqrt{s^2 + 1}} = \frac{s^2 + s\sqrt{s^2 + 1} + 1}{(s + \sqrt{s^2 + 1})\sqrt{s^2 + 1}} = 1 \end{aligned}$$

Since $|d\mathbf{x}/ds| = 1$, s is a natural parameter.

3.19. Prove Theorem 3.4(i): If $\mathbf{x} = \mathbf{x}(s)$ is a natural representation on I_s , then $|s_2 - s_1|$ is the length of the arc $\mathbf{x} = \mathbf{x}(s)$ between the points corresponding to $\mathbf{x}(s_1)$ and $\mathbf{x}(s_2)$.

$$\text{the length of the arc is } \int_{s_1}^{s_2} \left| \frac{d\mathbf{x}}{ds} \right| ds = \int_{s_1}^{s_2} 1 ds = s_2 - s_1 = |s_2 - s_1|.$$

$$\text{If } s_1 \leq s_2, \text{ the length of the arc is } \int_{s_1}^{s_2} \left| \frac{d\mathbf{x}}{ds} \right| ds = \int_{s_1}^{s_2} 1 ds = s_2 - s_1 = |s_2 - s_1|.$$

$$\text{If } s_1 > s_2, \text{ the length of the arc is } \int_{s_2}^{s_1} \left| \frac{d\mathbf{x}}{ds} \right| ds = \int_{s_2}^{s_1} 1 ds = s_1 - s_2 = |s_2 - s_1|.$$

If $s_1 > s_2$, the length of the arc is $\int_{s_2}^{s_1} |d\mathbf{x}/ds| ds = \int_{s_2}^{s_1} 1 ds = s_1 - s_2 = |s_2 - s_1|$.

If $s_1 > s_2$, the length of the arc is $\int_{s_2}^{s_1} |d\mathbf{x}/ds| ds = \int_{s_2}^{s_1} 1 ds = s_1 - s_2 = |s_2 - s_1|$.

3.20. Prove Theorem 3.4(ii): If $\mathbf{x} = \mathbf{x}(s)$ and $\mathbf{x} = \mathbf{x}^*(s^*)$ are natural representations of the same curve, then $s = \pm s^* + \text{constant}$.

$$\text{Then } \frac{d\mathbf{x}}{ds^*} = \frac{d\mathbf{x}}{ds} \frac{ds}{ds^*} \text{ and } \left| \frac{d\mathbf{x}}{ds^*} \right| = \left| \frac{d\mathbf{x}}{ds} \right| \left| \frac{ds}{ds^*} \right|.$$

$$\text{Let } s = s(s^*). \text{ Then } \frac{d\mathbf{x}}{ds^*} = \frac{d\mathbf{x}}{ds} \frac{ds}{ds^*} \text{ and } \left| \frac{d\mathbf{x}}{ds^*} \right| = 1 \text{ or } \frac{ds}{ds^*} = \pm 1 \text{ or } s = \pm s^* + \text{constant}.$$

$$\text{But } \left| \frac{d\mathbf{x}}{ds} \right| = \left| \frac{d\mathbf{x}}{ds^*} \right| = 1. \text{ Hence } \left| \frac{ds}{ds^*} \right| = 1 \text{ or } \frac{ds}{ds^*} = \pm 1 \text{ or } s = \pm s^* + \text{constant}.$$

3.21. Show that the arc $\mathbf{x} = t^2\mathbf{e}_1 + \sin t\mathbf{e}_2$, $0 \leq t \leq \pi/2$, is rectifiable.

0 = $t_0 < t_1 < t_2 < \dots < t_n = \pi/2$ and compute the

We consider an arbitrary subdivision

$$\begin{aligned} \text{length of the polygonal arc:} \\ s(P) = \sum_{i=1}^n |\mathbf{x}_i - \mathbf{x}_{i-1}| &= \sum_{i=1}^n |(t_i^2\mathbf{e}_1 + (\sin t_i)\mathbf{e}_2) - (t_{i-1}^2\mathbf{e}_1 + (\sin t_{i-1})\mathbf{e}_2)| \\ &= \sum_{i=1}^n [(t_i^2 - t_{i-1}^2)|\mathbf{e}_1| + |\sin t_i - \sin t_{i-1}||\mathbf{e}_2|] \end{aligned}$$

$\int_0^{\pi/2} \sqrt{4t^2 + 1} dt$

Using the mean value theorem, we have

$$\begin{aligned}s(P) &\leq \sum_i [(t_i - t_{i-1})(t_i + t_{i-1}) + |\cos \theta_i|(t_i - t_{i-1})] \\ &\leq \sum_i (t_i - t_{i-1})[t_i + t_{i-1} + |\cos \theta_i|] \quad t_{i-1} < \theta_i < t_i\end{aligned}$$

Since $|\cos \theta_i| \leq 1$ and $(t_i + t_{i-1}) \leq \pi$ for $0 \leq t_{i-1} < t_i \leq \pi/2$,

$$s(P) \leq (\pi + 1) \sum_i (t_i - t_{i-1}) \leq (\pi/2)(\pi + 1)$$

Since the $s(P)$ are bounded, the arc is rectifiable.

3.22. If $\mathbf{x} = \mathbf{f}(t)$, $a \leq t \leq b$, is a rectifiable arc, show that given an arbitrary $\delta > 0$ and $\epsilon > 0$, there exists a subdivision $a = t_0 < t_1 < \dots < t_n = b$ with polygonal approximation P such that

$$(i) \quad t_i - t_{i-1} < \delta, \quad i = 1, \dots, n \quad (ii) \quad |s - s(P)| < \epsilon$$

where s and $s(P)$ are lengths of $\mathbf{x} = \mathbf{f}(t)$ and P respectively.

Since s is the supremum of all possible $s(P)$, there exists a subdivision $a = t'_0 < t'_1 < \dots < t'_n = b$ with polygonal approximation P' such that $s(P') > s - \epsilon$. For otherwise, $s(P) \leq s - \epsilon$ for all $s(P)$ so that $s - \epsilon$ is an upper bound of the $s(P)$ less than the supremum s , which is impossible. Now, the above subdivision does not satisfy (i), a finer subdivision $a = t_0 < t_1 < \dots < t_n = b$ satisfying $(t_i - t_{i-1}) < \delta$ can be obtained by introducing additional points. But the new polygonal arc obtained this way satisfies $s(P) \leq s(P') \leq s$ and therefore also $|s - s(P)| < \epsilon$, as required.

3.23. Show that a regular arc $\mathbf{x} = \mathbf{f}(t)$, $a \leq t \leq b$, is rectifiable.

Consider an arbitrary subdivision $a = t_0 < t_1 < t_2 < \dots < t_n = b$. Then

$$\begin{aligned}s(P) &= \sum_i |\mathbf{x}_i - \mathbf{x}_{i-1}| = \sum_i |\mathbf{f}(t_i) - \mathbf{f}(t_{i-1})| \\ &= \sum_i |(f_1(t_i) - f_1(t_{i-1}))\mathbf{e}_1 + (f_2(t_i) - f_2(t_{i-1}))\mathbf{e}_2 + (f_3(t_i) - f_3(t_{i-1}))\mathbf{e}_3| \\ &\leq \sum_i [|f_1(t_i) - f_1(t_{i-1})| + |f_2(t_i) - f_2(t_{i-1})| + |f_3(t_i) - f_3(t_{i-1})|] \\ &\leq \sum_i [|f'_1(\theta_i)|(t_i - t_{i-1}) + |f'_2(\theta'_i)|(t_i - t_{i-1}) + |f'_3(\theta''_i)|(t_i - t_{i-1})]\end{aligned}$$

where we used the mean value theorem for the $f_i(t)$. Since the $f'_i(t)$ are continuous on the closed interval $a \leq t \leq b$, they are bounded on $a \leq t \leq b$, say by M_i . Hence

$$s(P) \leq (M_1 + M_2 + M_3) \sum_i (t_i - t_{i-1}) \leq (M_1 + M_2 + M_3)(b - a)$$

Thus the $s(P)$ are all bounded by $(M_1 + M_2 + M_3)(b - a)$ and so the arc is rectifiable.

3.24. Show that the length of a regular arc $\mathbf{x} = \mathbf{f}(t)$, $a \leq t \leq b$, is given by the integral $s = \int_a^b \left| \frac{d\mathbf{x}}{dt} \right| dt = \int_a^b |\mathbf{f}'(t)| dt$. This together with Problem 3.23 proves Theorem 3.3, page 51.

Let ϵ be arbitrary. Since the $f'_i(t)$, $i = 1, 2, 3$, are continuous on a closed interval $a \leq t \leq b$ they are uniformly continuous on the interval. Namely, there exists $\delta_1 > 0$ such that

$$(i) \quad |f'_i(t) - f'_i(t')| < \frac{\epsilon}{9(b-a)}, \quad i = 1, 2, 3$$

for all $|t - t'| < \delta_1$. Also, by definition of the integral, there exists a δ_2 such that for $|t_i - t_{i-1}| < \delta_2$ we have

$$(ii) \quad \left| \int_a^b |\mathbf{f}'(t)| dt - \sum_{i=1}^n |\mathbf{f}'(\theta_i)|(t_i - t_{i-1}) \right| < \epsilon/3 \quad t_{i-1} \leq \theta_i \leq t_i$$

Now let $\delta = \min(\delta_1, \delta_2)$. It follows from Problem 3.22 that there exists a subdivision $\alpha = t_0 < t_1 < \dots < t_n = b$ and polygonal approximation P such that $(t_i - t_{i-1}) < \delta$ and

(iii)

$$|s - s(P)| < \epsilon/3$$

Now consider the quantity

$$\begin{aligned} I &= \left| s - \int_a^b |f'(t)| dt \right| \leq |s - s(P)| + \left| s(P) - \int_a^b |f'(t)| dt \right| \\ &\leq \frac{\epsilon}{3} + \left| \sum_i |f(t_i) - f(t_{i-1})| - \int_a^b |f'(t)| dt \right| \\ &\leq \frac{\epsilon}{3} + \left| \sum_i [(f_1(t_i) - f_1(t_{i-1}))e_1 + (f_2(t_i) - f_2(t_{i-1}))e_2 + (f_3(t_i) - f_3(t_{i-1}))e_3] - \int_a^b |f'(t)| dt \right| \end{aligned}$$

It follows from the mean value theorem that

$$I \leq \epsilon/3 + \left| \sum_i [f'_1(t'_i)e_1 + f'_2(t'_i)e_2 + f'_3(t'_i)e_3](t_i - t_{i-1}) - \int_a^b |f'(t)| dt \right|$$

and, adding and subtracting $\sum_i |f'(t_i)|(t_i - t_{i-1})$,

$$\begin{aligned} I &\leq \epsilon/3 + \left| \sum_i f'(t_i)(t_i - t_{i-1}) - \int_a^b f'(t) dt \right| \\ &\quad + \left| \sum_i [|f'_1(t'_i)e_1 + f'_2(t'_i)e_2 + f'_3(t'_i)e_3] - |f'(t_i)|](t_i - t_{i-1}) \right| \end{aligned}$$

Using (ii) above and the inequality $||a| - |b|| \leq |a + b|$, we have

$$I < \epsilon/3 + \epsilon/3 + \sum [|f'_1(t'_i) - f'_1(t_i)| + |f'_2(t'_i) - f'_2(t_i)| + |f'_3(t'_i) - f'_3(t_i)|](t_i - t_{i-1})$$

Finally, using (i), we obtain

$$I < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3(b-a)} \sum_i (t_i - t_{i-1}) = \epsilon$$

Since ϵ is arbitrary,

$$I = \left| s - \int_a^b |f'(t)| dt \right| = 0 \quad \text{or} \quad s = \int_a^b |f'(t)| dt$$

Supplementary Problems

3.25. Show that the representation $\mathbf{x} = te_1 + (t^2 + 2)e_2 + (t^3 + t)e_3$

is regular for all t and sketch the projections on the x_1x_3 and x_1x_2 planes.

3.26. The conchoid of Nicomedes in polar coordinates is $r = \frac{a}{\cos \theta} + c$, $a \neq 0$, $c \neq 0$, $-\pi \leq \theta \leq \pi$. Sketch and find a representation in rectangular coordinates.

Ans. $x_1 = a + c \cos \theta$, $x_2 = a \tan \theta + c \sin \theta$

3.27. Find a representation of the intersection of the cylinders $x_3^2 = x_1$, $x_2^2 = 1 - x_1$ that does not involve radicals.

Hint. $x_2^2 + x_3^2 = 1$.

Ans. $x_1 = \cos^2 \theta$, $x_2 = \sin \theta$, $x_3 = \cos \theta$, $0 \leq \theta \leq 2\pi$

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- 3.28. The *hypocycloid* is the plane curve generated by a point P on the circumference of a circle C as C rolls without sliding on the *interior* of a fixed circle C_0 , as shown in Fig. 3-18. If C has radius r and C_0 is at the origin with radius r_0 and P is initially located at $(r_0, 0)$, find a representation of the hypocycloid.

Ans. $x_1 = (r_0 - r) \cos \theta + r \cos \left(\frac{r_0 - r}{r} \theta \right)$

$x_2 = (r_0 - r) \sin \theta - r \sin \left(\frac{r_0 - r}{r} \theta \right)$

- 3.29. If in the preceding problem $r_0 = 5$ and $r = 2$, the equation of the hypocycloid is

$x_1 = 3 \cos \theta + 2 \cos 3\theta/2, \quad x_2 = 3 \sin \theta - 2 \sin 3\theta/2$

Find the singular points and sketch.

Ans. $\theta = (4/5)n\pi, \quad n = 0, \pm 1, \dots$

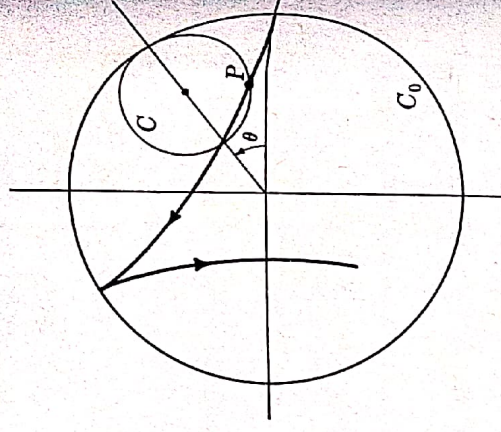


Fig. 3-18

- 3.30. Show that $\theta = 3t^5 + 10t^3 + 15t + 1$ is an allowable change of parameter for all t .

- 3.31. Find an allowable change of parameter which maps the interval $0 < t \leq 2$ onto $-\infty < \theta \leq 0$.

- 3.32. Compute the length of the arc $\mathbf{x} = e^t(\cos t)\mathbf{e}_1 + e^t(\sin t)\mathbf{e}_2 + e^t\mathbf{e}_3, \quad 0 \leq t \leq \pi$. Ans. $3(e^\pi - 1)$

- 3.33. Referring to Problem 3.28, find arc length as a function of θ along the hypocycloid.

$x_1 = (r_0 - r) \cos \theta + r \cos \left(\frac{r_0 - r}{r} \theta \right), \quad x_2 = (r_0 - r) \sin \theta - r \sin \left(\frac{r_0 - r}{r} \theta \right), \quad r_0 > r$

Ans. $s = \frac{4r(r_0 - r)}{r_0} [1 - \cos(r_0\theta/2r)]$

- 3.34. Show that $\mathbf{x} = t\mathbf{e}_1 + (\sin t)\mathbf{e}_2 + e^t\mathbf{e}_3, \quad -\infty < t < \infty$

and $\mathbf{x} = (\log t)\mathbf{e}_1 + \sin(\log t)\mathbf{e}_2 + t\mathbf{e}_3, \quad 0 < t < \infty$

are representations of the same oriented curve.

- 3.35. Show that an oriented regular curve has a representation defined on one of the following intervals: (i) $0 \leq t \leq 1$, (ii) $0 < t < 1$, (iii) $0 \leq t < 1$, (iv) $0 < t \leq 1$.

- 3.36. Suppose two regular representations $\mathbf{x} = \mathbf{x}(t)$ on I_t and $\mathbf{x} = \mathbf{x}^*(\theta)$ on I_θ are said to be equivalent if they represent the same oriented curve; i.e. if there exists an allowable change of parameter $t = t(\theta)$ such that $dt/d\theta > 0$, $t(I_\theta) = I_t$ and $\mathbf{x}(t(\theta)) = \mathbf{x}^*(\theta)$. Show that this is an equivalence relation on the set of regular representations. Thus a regular oriented curve is an equivalent class of regular representations related by allowable changes of parameter with a positive derivative.

- 3.37. Show that an arc segment $\mathbf{x} = \mathbf{x}(t)$ on I^* of a rectifiable arc $\mathbf{x} = \mathbf{x}(t)$ on I is rectifiable.

- 3.38. If $\mathbf{x} = \mathbf{x}(t)$ on $a \leq t \leq b$ is a rectifiable arc with length s and $a < t_0 < b$, show that the arc segments $\mathbf{x} = \mathbf{x}(t)$ on $a \leq t \leq t_0$ and $\mathbf{x} = \mathbf{x}(t)$ on $t_0 \leq t \leq b$ are both rectifiable with lengths s_1 and s_2 respectively and $s = s_1 + s_2$.

- 3.39. Prove Theorem 3.4(iii): If $\mathbf{x} = \mathbf{x}(s)$ is a natural representation of an oriented curve C and $\mathbf{x} = \mathbf{x}^*(t)$ is any other representation of C , then $ds/dt = |d\mathbf{x}/dt|$.

Chapter 4

Curvature and Torsion

INTRODUCTION

One of the basic problems in geometry is to determine exactly the geometric quantities which distinguish one figure from another. For example, line segments are uniquely determined by their lengths, circles by their radii, triangles by side-angle-side, etc. It turns out that this problem can be solved in general for sufficiently smooth regular curves. We will see that a regular curve is uniquely determined by two scalar quantities, called curvature and torsion, as functions of the natural parameter.

UNIT TANGENT VECTOR

Let $\mathbf{x} = \mathbf{x}(s)$ be a natural representation of a regular curve C . The derivative $d\mathbf{x}/ds = \dot{\mathbf{x}}(s)$ will be used to define the *tangent* direction to C at the point $\mathbf{x}(s)$. This agrees with our geometric intuition, since

$$\dot{\mathbf{x}}(s) = \lim_{\Delta s \rightarrow 0} \frac{\mathbf{x}(s + \Delta s) - \mathbf{x}(s)}{\Delta s}$$

and $\frac{\mathbf{x}(s + \Delta s) - \mathbf{x}(s)}{\Delta s}$ is a secant to C as shown in Fig. 4-1. The vector $\dot{\mathbf{x}}$ is also of unit length, for in a natural representation $|d\mathbf{x}/ds| = |\dot{\mathbf{x}}| = 1$.

If $\mathbf{x} = \mathbf{x}(s^*)$ is any other natural representation and $s = \pm s^* + \text{constant}$ and

$$\frac{d\mathbf{x}}{ds^*} = \frac{d\mathbf{x}}{ds} \frac{ds}{ds^*} = \pm \frac{d\mathbf{x}}{ds}$$

That is, $d\mathbf{x}/ds^*$ has the same or opposite sense as $d\mathbf{x}/ds$ depending on the orientation of $\mathbf{x} = \mathbf{x}(s^*)$. Thus $\dot{\mathbf{x}}$ is an *oriented* quantity. As shown in Fig. 4-1, it is in the direction of increasing s .

The vector $\dot{\mathbf{x}}(s)$ is called the *unit tangent vector* to the oriented curve $\mathbf{x} = \mathbf{x}(s)$ at $\mathbf{x}(s)$ and shall be denoted by $\mathbf{t} = \mathbf{t}(s) = \dot{\mathbf{x}}(s)$.

Example 4.1.

Along the helix $\mathbf{x} = a(\cos t)\mathbf{e}_1 + a(\sin t)\mathbf{e}_2 + bte_3$, $a, b \neq 0$, we have

$$\frac{d\mathbf{x}}{dt} = -a(\sin t)\mathbf{e}_1 + a(\cos t)\mathbf{e}_2 + b\mathbf{e}_3 \quad \text{and} \quad \left| \frac{d\mathbf{x}}{dt} \right| = (a^2 + b^2)^{1/2}$$

Then $\mathbf{t} = \frac{d\mathbf{x}}{ds} = \frac{d\mathbf{x}}{dt} \frac{dt}{ds} = \frac{d\mathbf{x}}{dt} \frac{ds}{dt} = \frac{d\mathbf{x}}{dt} \cdot \frac{1}{\left| \frac{d\mathbf{x}}{dt} \right|} = \frac{d\mathbf{x}}{dt} \cdot \frac{1}{(a^2 + b^2)^{1/2}} = \frac{1}{(a^2 + b^2)^{1/2}} (-a(\sin t)\mathbf{e}_1 + a(\cos t)\mathbf{e}_2 + b\mathbf{e}_3)$

where we used the fact that $ds/dt = |d\mathbf{x}/dt| = \sqrt{a^2 \cos^2 t + a^2 \sin^2 t + b^2} = \sqrt{a^2 + b^2}$ (Theorem 3.4). Observe that along the helix the unit tangent $\mathbf{t}$ makes a constant angle $\theta = \cos^{-1}(\mathbf{t} \cdot \mathbf{e}_3) = \cos^{-1} b/(a^2 + b^2)^{1/2}$ with the x_3 axis.

As in the case of the unit tangent, other geometric quantities along the curve will be defined in terms of a natural representation. However, by using the chain rule and the relation $ds/dt = |d\mathbf{x}/dt|$, these quantities can all be derived in terms of an arbitrary parameter, as in the example above.

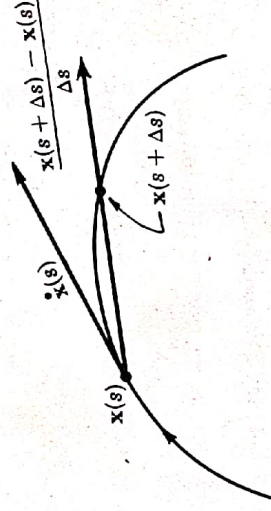


Fig. 4-1

in Fig. 4-1. The vector $\dot{\mathbf{x}}$ is also of unit length, for in a natural representation $|d\mathbf{x}/ds| = |\dot{\mathbf{x}}| = 1$.

If $\mathbf{x} = \mathbf{x}(t)$ is an arbitrary representation of C with the same orientation as $\mathbf{x} = \mathbf{x}(s)$, then

$$\mathbf{x}' = \frac{d\mathbf{x}}{dt} = \frac{d\mathbf{x}}{ds} \frac{ds}{dt} = \mathbf{t} \left| \frac{ds}{dt} \right| = \mathbf{t} |\mathbf{x}'|$$

where again we used $ds/dt = |d\mathbf{x}/dt|$. Thus as we expect, the derivative $\mathbf{x}'$ has the same direction as $\mathbf{t}$, namely, it is also a tangent vector to the curve and we have the formula

$$\mathbf{t} = \mathbf{x}'/|\mathbf{x}'| \quad (4.1)$$

TANGENT LINE AND NORMAL PLANE

The straight line through a point $\mathbf{x}$ on a regular curve C parallel to the tangent vectors at $\mathbf{x}$ is called the *tangent line* to C at $\mathbf{x}$. See Fig. 4-2. It follows from equation (2.1), page 21, that the tangent line at the point $\mathbf{x}_0 = \mathbf{x}(t_0)$ is given by

$$\mathbf{x} = \mathbf{x}_0 + k\mathbf{t}_0, \quad -\infty < k < \infty$$

where $\mathbf{t}_0 = \mathbf{t}(t_0)$ is a unit tangent at $\mathbf{x}_0$.

The plane through $\mathbf{x}$ orthogonal to the tangent line at $\mathbf{x}$ is called the *normal plane* to C at $\mathbf{x}$. It follows from equation (2.5), page 21, the normal plane at $\mathbf{x}_0$ is given by

$$(\mathbf{x} - \mathbf{x}_0) \cdot \mathbf{t}_0 = 0$$

It is convenient to introduce a second variable, say $\mathbf{y}$, to denote a generic point on the figure related to $\mathbf{x} = \mathbf{x}(t)$. Using this, we can express the equation of the tangent line at an arbitrary point $\mathbf{x}$ on C as

$$\mathbf{y} = \mathbf{x} + k\mathbf{t}, \quad -\infty < k < \infty \quad (4.2)$$

and the normal plane at $\mathbf{x}$ as

$$(\mathbf{y} - \mathbf{x}) \cdot \mathbf{t} = 0 \quad (4.3)$$

Finally we note that $\mathbf{x}'$ is parallel to $\mathbf{t}$, so that the tangent line and normal plane are also given by

$$\mathbf{y} = \mathbf{x} + k\mathbf{x}', \quad -\infty < k < \infty$$

and

$$(\mathbf{y} - \mathbf{x}) \cdot \mathbf{x}' = 0$$

respectively.

* Example 4.2.

The tangent line to the curve $\mathbf{x} = t\mathbf{e}_1 + t^2\mathbf{e}_2 + t^3\mathbf{e}_3$ at $t = 1$ is

$$\mathbf{y} = \mathbf{x}(1) + k\mathbf{x}'(1) \quad \text{or} \quad \mathbf{y} = (1+k)\mathbf{e}_1 + (1+2k)\mathbf{e}_2 + (1+3k)\mathbf{e}_3, \quad -\infty < k < \infty$$

The normal plane at $t = 1$ is

$$(\mathbf{y} - \mathbf{x}(1)) \cdot \mathbf{x}'(1) = 0 \quad \text{or} \quad (y_1 - 1) + (y_2 - 1)2 + (y_3 - 1)3 = 0$$

$$\text{or } y_1 + 2y_2 + 3y_3 = 6.$$

CURVATURE

We now assume that $\mathbf{x} = \mathbf{x}(s)$ is a regular curve of class ≥ 2 . Then the tangent vector $\mathbf{t} = \mathbf{t}(s) = \dot{\mathbf{x}}(s)$ is of class C^1 , and we can consider its derivative

$$d\mathbf{t}/ds = \dot{\mathbf{t}}(s) = \ddot{\mathbf{x}}(s)$$

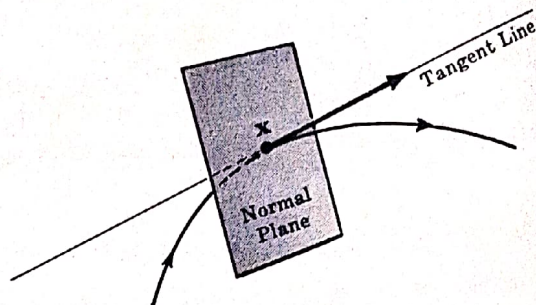


Fig. 4-2

Whereas the sense of the unit tangent $\mathbf{t}$ depends upon the orientation of C , $\dot{\mathbf{t}}$ is independent of orientation. For let $\mathbf{x} = \mathbf{x}(s^*)$ be any other natural representation of C with, say, unit tangent $\mathbf{t}^* = d\mathbf{x}/ds^*$. Then $s = \pm s^* + \text{constant}$ and

$$\frac{d\mathbf{t}^*}{ds^*} = \frac{d}{ds^*} \left(\frac{d\mathbf{x}}{ds^*} \right) = \frac{d}{ds^*} \left(\pm \frac{d\mathbf{x}}{ds} \right) = \pm \frac{d}{ds} \left(\frac{d\mathbf{x}}{ds} \right) \frac{ds}{ds^*} = (\pm 1)^2 \frac{d}{ds} \left(\frac{d\mathbf{x}}{ds} \right) = \frac{d\mathbf{t}}{ds}$$

Thus $\dot{\mathbf{t}}$ is independent of orientation.

The vector $\dot{\mathbf{t}}(s)$ is called the *curvature vector* on C at the point $\mathbf{x}(s)$ and is denoted by $\mathbf{k} = \mathbf{k}(s) = \dot{\mathbf{t}}(s)$.

Since $\mathbf{t}$ is a unit vector, it follows from Theorem 2.7, page 29, that $\mathbf{k} = \dot{\mathbf{t}}$ is orthogonal to $\mathbf{t}$ and hence parallel to the normal plane. When different from zero, it is in the direction in which the curve is turning, as indicated in Fig. 4-3.

The magnitude of the curvature vector is denoted by

$$|\kappa| = |\mathbf{k}(s)| \quad (4.4)$$

and is called the *curvature* of C at $\mathbf{x}(s)$. The reciprocal of the curvature is denoted by

$$\rho = \frac{1}{|\kappa|} = \frac{1}{|\mathbf{k}(s)|} \quad (4.5)$$

and is called the *radius of curvature* at $\mathbf{x}(s)$.

A point on C where the curvature vector $\mathbf{k} = 0$ is called a *point of inflection*. Thus at a point of inflection the curvature $|\kappa|$ is zero and the radius of curvature ρ is infinite.

In Problem 4.9, page 73, we prove that the curvature is equal to the rate of change of the direction of the tangent with respect to arc length. Thus along a curve which has a rapidly changing tangent direction with respect to arc length, such as a circle with a small radius, the curvature is relatively large, or, equivalently, the radius of curvature is small.

Example 4.3.

Along the circle of radius a , $\mathbf{x} = a(\cos t)\mathbf{e}_1 + a(\sin t)\mathbf{e}_2$, $a > 0$, we have

$$\begin{aligned} \frac{d\mathbf{x}}{dt} &= -a(\sin t)\mathbf{e}_1 + a(\cos t)\mathbf{e}_2, \quad \left| \frac{d\mathbf{x}}{dt} \right| = a \\ \mathbf{t} &= \frac{d\mathbf{x}}{dt} / \left| \frac{d\mathbf{x}}{dt} \right| = -(\sin t)\mathbf{e}_1 + (\cos t)\mathbf{e}_2 \end{aligned}$$

$$\text{and } \mathbf{k} = \dot{\mathbf{t}} = \frac{d\mathbf{t}}{ds} = \frac{d\mathbf{t}}{dt} \frac{dt}{ds} = \frac{d\mathbf{t}}{dt} / \left| \frac{d\mathbf{x}}{dt} \right| = -\frac{1}{a}((\cos t)\mathbf{e}_1 + (\sin t)\mathbf{e}_2)$$

Note that $\mathbf{k}$ is directed towards the origin. The curvature is constant, equal to $|\kappa| = |\mathbf{k}| = 1/a$ and the radius of curvature is $\rho = 1/|\kappa| = a$. Hence as we expect, the radius of curvature of a circle is simply its radius.

Example 4.4.

Along the helix $\mathbf{x} = a(\cos t)\mathbf{e}_1 + a(\sin t)\mathbf{e}_2 + bte_3$, $a > 0$, $b \neq 0$, we have

$$\begin{aligned} \frac{d\mathbf{x}}{dt} &= -a(\sin t)\mathbf{e}_1 + a(\cos t)\mathbf{e}_2 + b\mathbf{e}_3, \quad \left| \frac{d\mathbf{x}}{dt} \right| = (a^2 + b^2)^{1/2} \\ \mathbf{t} &= \frac{d\mathbf{x}}{dt} / \left| \frac{d\mathbf{x}}{dt} \right| = (a^2 + b^2)^{-1/2}(-a(\sin t)\mathbf{e}_1 + a(\cos t)\mathbf{e}_2 + b\mathbf{e}_3) \end{aligned}$$

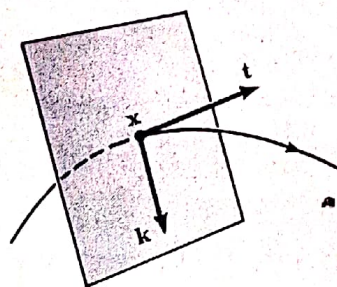


Fig. 4-3

and

$$\begin{aligned}\mathbf{k} &= \dot{\mathbf{t}} = \frac{d\mathbf{t}}{dt} / \left| \frac{d\mathbf{x}}{dt} \right| \\ &= (a^2 + b^2)^{-1/2} (-a(\cos t)\mathbf{e}_1 \\ &\quad - a(\sin t)\mathbf{e}_2) / (a^2 + b^2)^{1/2} \\ &= -\frac{a}{a^2 + b^2} ((\cos t)\mathbf{e}_1 + (\sin t)\mathbf{e}_2)\end{aligned}$$

Observe that $\mathbf{k}$ is parallel to the x_1x_2 plane and directed toward the origin, as shown in Fig. 4-4. The curvature is constant, equal to $|\kappa| = |\mathbf{k}| = a/(a^2 + b^2)$.

If the curvature is identically zero along a curve C , i.e. if $|\mathbf{k}| \equiv 0$, then $\dot{\mathbf{t}} \equiv 0$ and, integrating,

$$\mathbf{t} = \mathbf{a}, \quad \mathbf{a} = \text{constant} \neq 0$$

Since $\mathbf{t} = \dot{\mathbf{x}}$, we have, integrating again,

$$\mathbf{x} = \mathbf{a}t + \mathbf{b}, \quad \mathbf{b} = \text{constant}$$

that is, C is a straight line through (b_1, b_2, b_3) parallel to $\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3$. Conversely, if C is the straight line

$$\mathbf{x} = \mathbf{a}t + \mathbf{b}, \quad \mathbf{a} \neq 0$$

then

$$\mathbf{t} = \frac{d\mathbf{x}}{dt} / \left| \frac{d\mathbf{x}}{dt} \right| = \frac{\mathbf{a}}{|\mathbf{a}|} \quad \text{and} \quad |\mathbf{k}| = |\dot{\mathbf{t}}| = 0$$

Thus we have

Theorem 4.1. A regular curve of class ≥ 2 is a straight line if and only if its curvature is identically zero.

A formula can be derived which expresses the curvature directly in term of the derivatives of an arbitrary representation. Namely, in Problem 4.7, page 73, we prove

Theorem 4.2. If $\mathbf{x} = \mathbf{x}(t)$ is an arbitrary representation of a curve of class ≥ 2 , then

$$|\kappa| = |\mathbf{x}' \times \mathbf{x}''| / |\mathbf{x}'|^3$$

PRINCIPAL NORMAL UNIT VECTOR

Since C is of class ≥ 2 , the curvature vector $\mathbf{k} = \dot{\mathbf{t}} = \ddot{\mathbf{x}}$ varies continuously along C however, the unit vector in the direction of $\mathbf{k}$,

$$\mathbf{u}_k = \mathbf{k}/|\mathbf{k}|$$

is not defined where $\mathbf{k} = 0$, and may jump, as in the examples below. Thus we are led to consider not $\mathbf{u}_k$ itself but a unit vector parallel to $\mathbf{k}$ with its sense chosen arbitrarily but in such a way as to be continuous along C wherever possible. This vector shall be denoted by $\mathbf{n} = \mathbf{n}(s)$ and is called the *principal normal unit vector* to C at the point $\mathbf{x}(s)$.

Note that if there are no points of inflection on C , i.e. if $\mathbf{k}(s) \neq 0$ for all s , we can simply choose

$$\mathbf{n} = \mathbf{k}(s)/|\mathbf{k}(s)|$$

the unit vector in the direction of $\mathbf{k}$. Note also that along a straight line we have $\mathbf{k} \equiv 0$, so that the principal normal unit vector $\mathbf{n}$ is undetermined.

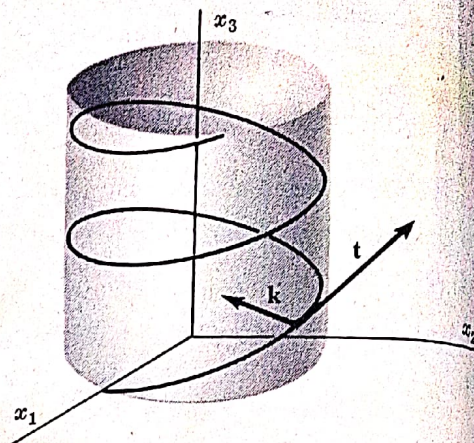


Fig. 4-4

Having selected $\mathbf{n}(s)$, there will be a continuous function $\kappa(s)$ along C such that

$$\mathbf{k}(s) = \kappa(s) \mathbf{n}(s) \quad (4.6)$$

At a point on the curve where $\mathbf{n}$ has the same direction as $\mathbf{k}$, we have $\kappa = |\mathbf{k}|$; where $\mathbf{n}$ is opposite to $\mathbf{k}$, $\kappa = -|\mathbf{k}|$; at a point of inflection, $\mathbf{k} = 0$ and $\kappa = 0$.

The quantity $\kappa(s)$ defined by equation (4.6) is also called the *curvature* of C at the point $\mathbf{x}(s)$. Note, however, since the sense of $\mathbf{n}$ is initially arbitrary, the function $\kappa(s)$ is determined only within a sign; and locally only its absolute value $|\kappa| = |\mathbf{k}|$, the curvature as defined previously, is an intrinsic property of the curve.

If we multiply (4.6) by $\mathbf{n}$ and use $\mathbf{n} \cdot \mathbf{n} = |\mathbf{n}|^2 = 1$, we have the formula

$$\kappa = \mathbf{k}(s) \cdot \mathbf{n}(s) \quad (4.7)$$

Example 1.3.

Along the third degree curve $\mathbf{x} = t\mathbf{e}_1 + \frac{1}{3}t^3\mathbf{e}_2$ shown in Fig. 4.5,

$$\frac{d\mathbf{x}}{dt} = \mathbf{e}_1 + t^2\mathbf{e}_2, \quad \left| \frac{d\mathbf{x}}{dt} \right| = (1 + t^4)^{1/2}, \quad \mathbf{t} = \frac{d\mathbf{x}}{ds} / \left| \frac{d\mathbf{x}}{ds} \right| = (1 + t^4)^{-1/2}(\mathbf{e}_1 + t^2\mathbf{e}_2)$$

and

$$\mathbf{k} = \dot{\mathbf{t}} = \frac{d\mathbf{t}}{ds} = \frac{d\mathbf{t}}{dt} / \left| \frac{d\mathbf{x}}{dt} \right| = -2t(1 + t^4)^{-3/2}(\mathbf{e}_1 - t^2\mathbf{e}_2)$$

The direction of $\mathbf{k}$ is shown in Fig. 4.5(a).

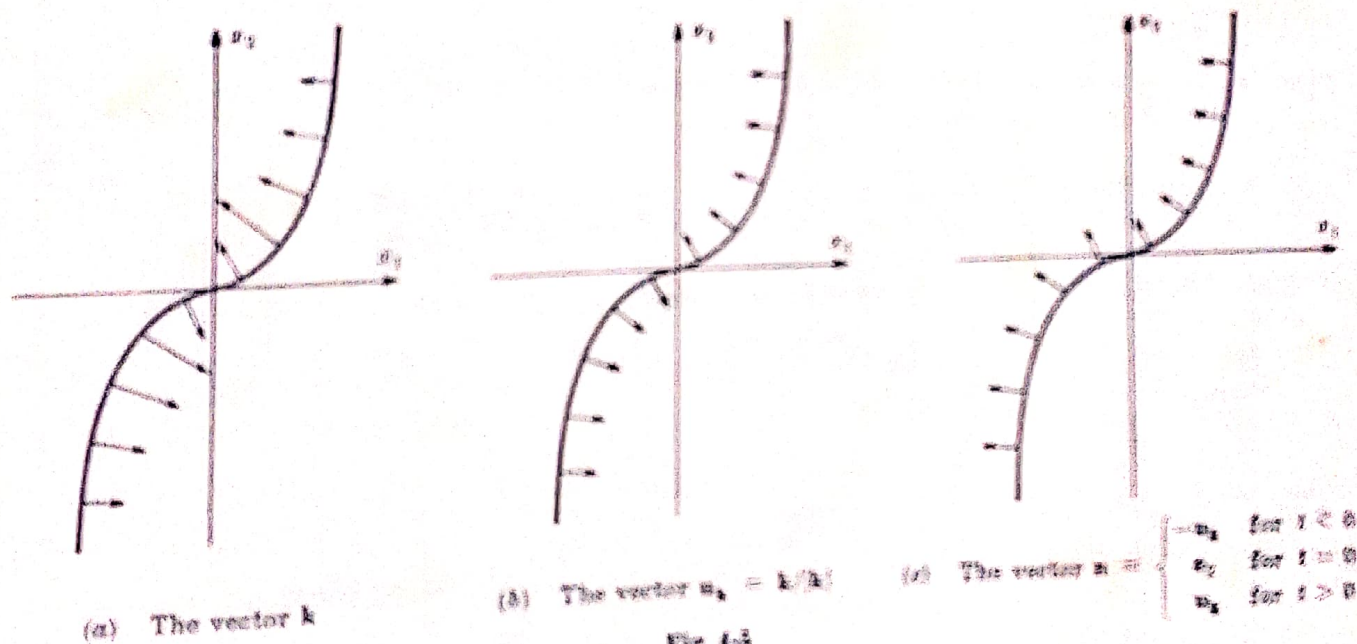


Fig. 4.5

At $t = 0$, $\mathbf{k} = 0$ and we have a point of inflection. Here $\mathbf{u}_k$ has a jump in sense as shown in Fig. 4.5(b). For, the limit of $\mathbf{u}_k$ as t approaches 0 through positive t is

$$\lim_{t \rightarrow 0^+} \mathbf{u}_k = \lim_{t \rightarrow 0^+} \frac{\mathbf{k}}{|\mathbf{k}|} = \lim_{t \rightarrow 0^+} \frac{-t}{t(1 + t^4)^{3/2}} (t^2\mathbf{e}_1 - \mathbf{e}_2) = \lim_{t \rightarrow 0^+} \frac{-(t^2\mathbf{e}_1 - \mathbf{e}_2)}{(1 + t^4)^{3/2}} = \mathbf{e}_2$$

whereas the limit of $\mathbf{u}_k$ as t approaches 0 through negative t is

$$\lim_{t \rightarrow 0^-} \mathbf{u}_k = \lim_{t \rightarrow 0^-} \frac{\mathbf{k}}{|\mathbf{k}|} = \lim_{t \rightarrow 0^-} \frac{-t}{t(1 + t^4)^{3/2}} (t^2\mathbf{e}_1 - \mathbf{e}_2) = \lim_{t \rightarrow 0^-} \frac{t^2\mathbf{e}_1 - \mathbf{e}_2}{(1 + t^4)^{3/2}} = -\mathbf{e}_2$$

where we used the fact that $\frac{t}{|t|} = \begin{cases} 1 & \text{for } t > 0 \\ -1 & \text{for } t < 0 \end{cases}$. If we choose

$$\mathbf{n} = \begin{cases} -\mathbf{k}/|\mathbf{k}| & \text{for } t < 0 \\ \mathbf{e}_2 & \text{for } t = 0 \\ \mathbf{k}/|\mathbf{k}| & \text{for } t > 0 \end{cases} = -(1+t^4)^{-1/2}(t^2\mathbf{e}_1 - \mathbf{e}_2)$$

$\mathbf{n}$ will vary continuously along the curve as shown in Fig. 4-5(c). For this $\mathbf{n}$ we have, from equation (4.7)

$$\kappa = \mathbf{k} \cdot \mathbf{n} = [-2t(1+t^4)^{-2}(t^2\mathbf{e}_1 - \mathbf{e}_2)] \cdot [-(1+t^4)^{-1/2}(t^2\mathbf{e}_1 - \mathbf{e}_2)] = 2t(1+t^4)^{-3/2}$$

Example 4.6.

We consider the curve of class C^∞ (see Example 3.13, page 49):

$$\mathbf{x} = \begin{cases} t\mathbf{e}_1 + e^{-1/t^2}\mathbf{e}_3 & \text{for } t < 0 \\ 0 & \text{for } t = 0 \\ t\mathbf{e}_1 + e^{-1/t^2}\mathbf{e}_2 & \text{for } t > 0 \end{cases}$$

As shown in Fig. 4-6, the curve lies in the x_1x_3 plane for $t < 0$ and in the x_1x_2 plane for $t > 0$. It follows that $\mathbf{k}$ lies in the x_1x_3 plane for $t < 0$ and in the x_1x_2 plane for $t > 0$. Here it is impossible to define $\mathbf{n}$ so that it is continuous at $t = 0$, since $\mathbf{k}$ jumps from the x_1x_3 plane to the x_1x_2 plane.

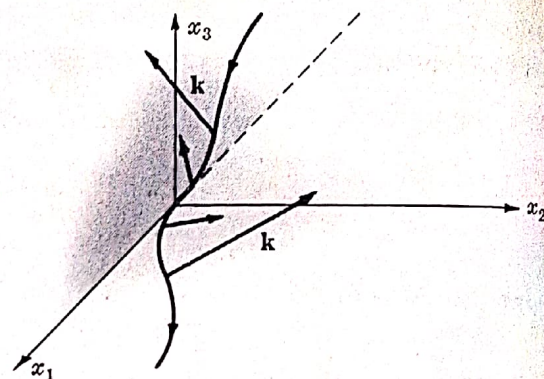


Fig. 4-6

As shown in the example above, even a curve of class C^∞ may not have a definite principal normal at a point of inflection. However, if the curve is analytic, a continuous principal normal will always exist. Namely, in Problem 4.15, page 75, we prove

Theorem 4.3. An analytic curve which is not a straight line will have a definite continuous principal normal unit vector in a neighborhood of a point of inflection.

PRINCIPAL NORMAL LINE AND OSCULATING PLANE

The straight line passing through a point $\mathbf{x}$ on a curve C parallel to the principal unit normal as shown in Fig. 4-7 is called the *principal normal line* to C at $\mathbf{x}$. It follows that the equation of the principal normal line at $\mathbf{x}$ is

$$\mathbf{y} = \mathbf{x} + k\mathbf{n}, \quad -\infty < k < \infty \quad (4.8)$$

The plane parallel to the unit tangent and the principal unit normal is called the *osculating plane* to C at $\mathbf{x}$. It follows from Example 2.3, page 22, that the equation of the osculating plane at $\mathbf{x}$ is given by the triple scalar product

$$[(\mathbf{y} - \mathbf{x})\mathbf{t}\mathbf{n}] = 0 \quad (4.9)$$

If we use the fact that $\mathbf{t} = \dot{\mathbf{x}}$ and that $\dot{\mathbf{t}} = \ddot{\mathbf{x}}$ is parallel to $\mathbf{n}$, then at a point where $\kappa \neq 0$, the equation of the osculating plane is also given by

$$[(\mathbf{y} - \mathbf{x})\ddot{\mathbf{x}}\dot{\mathbf{x}}] = 0 \quad (4.10)$$

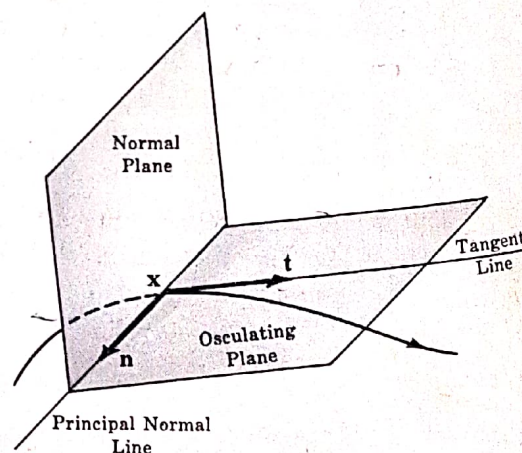


Fig. 4-7

We recall that the tangent line at a point on a curve can be defined as the limiting position of a line passing through two neighboring points on the curve as the two points

(22)

approach the given point. In this way a line is obtained that in a sense best fits the curve at a point. Similarly, the osculating plane at a point can be defined as the limiting position of a plane passing through three neighboring points on the curve as the points approach the given point. The tangent line and osculating plane are examples of geometric figures which have a certain order of *contact* with the curve. The theory of contact between curves and surfaces will be considered in the next chapter and the osculating plane will be considered again from this point of view.

Example 4.7.

Consider the helix $\mathbf{x} = (\cos t)\mathbf{e}_1 + (\sin t)\mathbf{e}_2 + t\mathbf{e}_3$.

$$\mathbf{x}' = (-\sin t)\mathbf{e}_1 + (\cos t)\mathbf{e}_2 + \mathbf{e}_3, \quad |\mathbf{x}'| = \sqrt{2}$$

$$\mathbf{t} = \mathbf{x}'/|\mathbf{x}'| = (1/\sqrt{2})(-\sin t)\mathbf{e}_1 + (\cos t)\mathbf{e}_2 + \mathbf{e}_3$$

and

$$\mathbf{k} = \dot{\mathbf{t}} = \mathbf{t}'/|\mathbf{x}'| = -(\frac{1}{2})(\cos t)\mathbf{e}_1 + (\sin t)\mathbf{e}_2$$

and, since $\mathbf{k} \neq 0$ for all t ,

$$\mathbf{n} = \mathbf{k}/|\mathbf{k}| = -(\cos t)\mathbf{e}_1 + (\sin t)\mathbf{e}_2$$

The equation of the principal normal at $t = \pi/2$ is

$$\mathbf{y} = \mathbf{x}(\pi/2) + k\mathbf{n}(\pi/2) \quad \text{or} \quad \mathbf{y} = (1-k)\mathbf{e}_2 + \pi/2\mathbf{e}_3, \quad -\infty < k < \infty$$

and the equation of the osculating plane at $t = \pi/2$ is

$$[(\mathbf{y} - \mathbf{x}(\pi/2)) \cdot \mathbf{t}(\pi/2)\mathbf{n}(\pi/2)] = 0$$

or

$$\det \begin{pmatrix} y_1 & -1/\sqrt{2} & 0 \\ y_2 - 1 & 0 & -1 \\ y_3 - \pi/2 & 1/\sqrt{2} & 0 \end{pmatrix} = 0 \quad \text{or} \quad y_1 + y_3 = \pi/2$$

BINORMAL. MOVING TRIHEDRON

Let $\mathbf{x} = \mathbf{x}(s)$ be a regular curve C of class ≥ 2 along which $\mathbf{n}$ is continuous. Then at each point on C we have two orthogonal and continuous unit vectors: the unit tangent $\mathbf{t}$ and the unit principal normal $\mathbf{n}$. Now consider the vector

$$\mathbf{b}(s) = \mathbf{t}(s) \times \mathbf{n}(s)$$

We observe that $\mathbf{b}$ is continuous and of unit length, and that $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ forms a right-handed orthonormal triplet as shown in Fig. 4-8. The vector $\mathbf{b}(s)$ is called the *unit binormal vector* to C at the point $\mathbf{x}(s)$ and the triplet $(\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s))$ is called the *moving trihedron* of C .

The straight line through $\mathbf{x}$ parallel to $\mathbf{b}$ is called the *binormal line* to C at $\mathbf{x}$. It follows that the equation of the binormal at $\mathbf{x}$ is

$$\mathbf{y} = \mathbf{x} + k\mathbf{b}, \quad -\infty < k < \infty \quad (4.11)$$

The plane through a point $\mathbf{x}$ on C parallel to $\mathbf{t}$ and $\mathbf{b}$ is called the *rectifying plane* at $\mathbf{x}$. The equation of the rectifying plane at $\mathbf{x}$ is

$$(\mathbf{y} - \mathbf{x}) \cdot \mathbf{n} = 0 \quad (4.12)$$

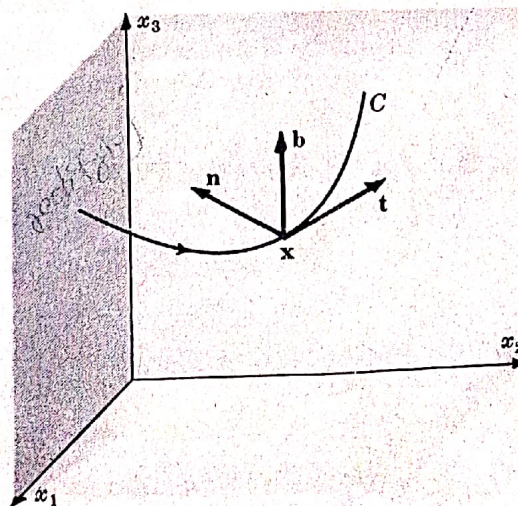


Fig. 4-8

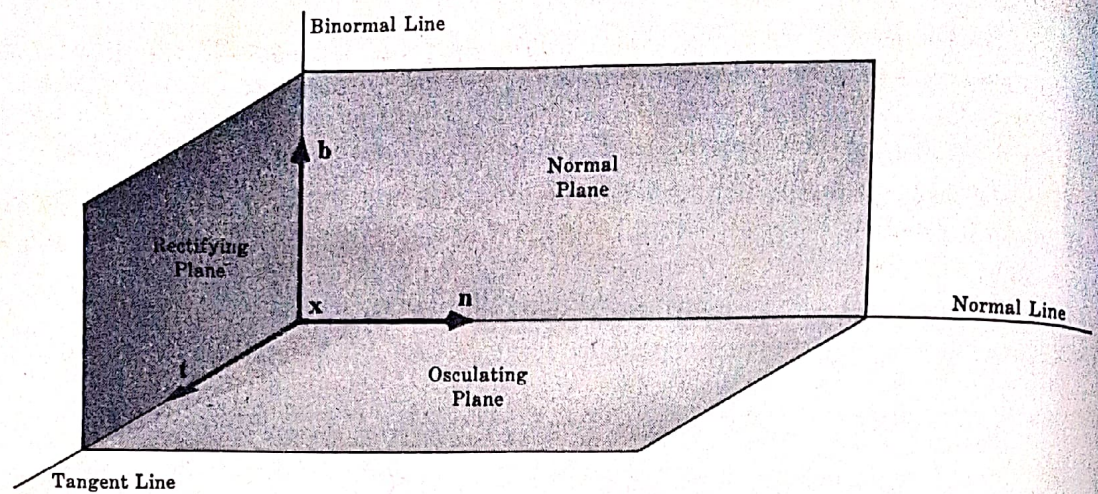


Fig. 4-9

Thus at each point x on C we have the following three characteristic lines and plane

Tangent line: $y = x + kt$

Principal normal line: $y = x + kn$

Binormal line: $y = x + kb$

Normal plane: $(y - x) \cdot t = 0$

Rectifying plane: $(y - x) \cdot n = 0$

Osculating plane: $(y - x) \cdot b = 0$

Example 4.8.

Referring to the helix in Example 4.4, we have

$$x = a(\cos t)e_1 + a(\sin t)e_2 + bte_3, \quad a > 0, \quad b \neq 0$$

$$t = (a^2 + b^2)^{-1/2}(-a(\sin t)e_1 + a(\cos t)e_2 + be_3)$$

$$k = -\frac{a}{a^2 + b^2}((\cos t)e_1 + (\sin t)e_2), \quad n = \frac{k}{|k|} = -((\cos t)e_1 + (\sin t)e_2)$$

$$b = t \times n = \det \begin{pmatrix} e_1 & -a(a^2 + b^2)^{-1/2} \sin t & -\cos t \\ e_2 & a(a^2 + b^2)^{-1/2} \cos t & -\sin t \\ e_3 & b(a^2 + b^2)^{-1/2} & 0 \end{pmatrix}$$

$$= (a^2 + b^2)^{-1/2}(b(\sin t)e_1 - b(\cos t)e_2 + ae_3)$$

The equation of the binormal line at $t = t_0$ is

$$y = x(t_0) + kb(t_0)$$

or

$$y = (a \cos t_0 + kb(a^2 + b^2)^{-1/2} \sin t_0)e_1 + (a \sin t_0 - kb(a^2 + b^2)^{-1/2} \cos t_0)e_2 + (bt_0 + ak(a^2 + b^2)^{-1/2})e_3, \quad -\infty < k < \infty$$

Or, if we introduce the change in parameter $\theta = k(a^2 + b^2)^{-1/2}$,

$$y = (a \cos t_0 + \theta b \sin t_0)e_1 + (a \sin t_0 - \theta b \cos t_0)e_2 + (bt_0 + a\theta)e_3, \quad -\infty < \theta < \infty$$

The equation of the rectifying plane at $t = t_0$ is

$$(y - x(t_0)) \cdot n(t_0) = 0$$

or

$$(y_1 - a \cos t_0)(-\cos t_0) + (y_2 - a \sin t_0)(-\sin t_0) = 0$$

or

$$y_1 \cos t_0 + y_2 \sin t_0 = a$$

Observe that rectifying planes are parallel to the x_3 axis.

TORSION

We now suppose that $\mathbf{x} = \mathbf{x}(s)$ is a regular curve of class ≥ 3 along which $\mathbf{n}(s)$ is of class C^1 . Then we can differentiate $\mathbf{b}(s) = \mathbf{t}(s) \times \mathbf{n}(s)$, obtaining

$$\dot{\mathbf{b}}(s) = \dot{\mathbf{t}}(s) \times \mathbf{n}(s) + \mathbf{t}(s) \times \dot{\mathbf{n}}(s) = \kappa(s)[\mathbf{n}(s) \times \mathbf{n}(s)] + \mathbf{t}(s) \times \dot{\mathbf{n}}(s) = \mathbf{t}(s) \times \dot{\mathbf{n}}(s) \quad (4.13)$$

where we used equation (4.6) and the fact that $\mathbf{a} \times \mathbf{a} = \mathbf{0}$ for all $\mathbf{a}$. Since $\mathbf{n}$ is a unit vector, $\dot{\mathbf{n}}$ is orthogonal to $\mathbf{n}$ and therefore is parallel to the rectifying plane. It follows that $\dot{\mathbf{n}}$ is a linear combination of $\mathbf{t}$ and $\mathbf{b}$, say

$$\dot{\mathbf{n}}(s) = \mu(s)\mathbf{t}(s) + \tau(s)\mathbf{b}(s).$$

Substituting into (4.13),

$$\dot{\mathbf{b}}(s) = \mathbf{t}(s) \times [\mu(s)\mathbf{t}(s) + \tau(s)\mathbf{b}(s)] = \tau(s)[\mathbf{t}(s) \times \mathbf{b}(s)]$$

or

$$\dot{\mathbf{b}}(s) = -\tau(s)\mathbf{n}(s) \quad (4.14)$$

where we use the fact that $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ is a right-handed orthonormal triplet and hence $\mathbf{t} \times \mathbf{b} = -\mathbf{n}$.

The continuous function $\tau(s)$ defined by (4.14) is called the *second curvature* or *torsion* of C at $\mathbf{x}(s)$. Note that if we take the scalar product of (4.14) with $\mathbf{n}$, we obtain the formula

$$\tau = -\dot{\mathbf{b}}(s) \cdot \mathbf{n}(s) \quad (4.15)$$

Note that the sign of τ is independent of the sense of $\mathbf{n}$ and the orientation of C , and hence is an intrinsic property of the curve. For suppose first that we change the sense of $\mathbf{n}$, i.e. suppose $\mathbf{n}^* = -\mathbf{n}$, then $\mathbf{b}^* = \mathbf{t} \times \mathbf{n}^* = \mathbf{t} \times (-\mathbf{n}) = -\mathbf{b}$ and, from equation (4.15),

$$\tau^* = -\dot{\mathbf{b}}^* \cdot \mathbf{n}^* = -(-\dot{\mathbf{b}}) \cdot (-\mathbf{n}) = -\dot{\mathbf{b}} \cdot \mathbf{n} = \tau$$

Thus τ is independent of the sense of $\mathbf{n}$. Next, suppose we introduce a change in orientation; that is, suppose $s = -s^* + \text{constant}$. Then $\mathbf{t}^* = -\mathbf{t}$. Also,

$$\mathbf{b}^* = \mathbf{t}^* \times \mathbf{n} = -(\mathbf{t} \times \mathbf{n}) = -\mathbf{b}$$

and

$$\frac{d\mathbf{b}^*}{ds^*} = \frac{d\mathbf{b}^*}{ds} \frac{ds}{ds^*} = -\frac{d\mathbf{b}}{ds}(-1) = \frac{d\mathbf{b}}{ds}$$

so that again

$$\tau^* = -\frac{d\mathbf{b}^*}{ds^*} \cdot \mathbf{n} = -\frac{d\mathbf{b}}{ds} \cdot \mathbf{n} = \tau$$

which is the required result.

Example 4.9.

We consider again the helix

$$\mathbf{x} = a(\cos t)\mathbf{e}_1 + a(\sin t)\mathbf{e}_2 + bte_3, \quad a > 0, b \neq 0$$

Referring to Example 4.8, we have

$$\begin{aligned} \mathbf{b} &= (a^2 + b^2)^{-1/2}(b(\sin t)\mathbf{e}_1 - b(\cos t)\mathbf{e}_2 + ae_3) \\ \dot{\mathbf{b}} &= \frac{d\mathbf{b}}{ds} = \frac{d\mathbf{b}}{dt} \bigg/ \left| \frac{d\mathbf{x}}{dt} \right| = (a^2 + b^2)^{-1}(b(\cos t)\mathbf{e}_1 + b(\sin t)\mathbf{e}_2) \end{aligned}$$

The torsion is constant, equal to

$$\tau = -\dot{\mathbf{b}} \cdot \mathbf{n} = -(a^2 + b^2)^{-1}(b(\cos t)\mathbf{e}_1 + b(\sin t)\mathbf{e}_2) \cdot ((-\cos t)\mathbf{e}_1 - (\sin t)\mathbf{e}_2) = b/(a^2 + b^2)$$

Note that if $b > 0$ (so that $\tau > 0$) the helix is a right-handed curve as shown in Fig. 4-10(a) below. If $b < 0$ (so that $\tau < 0$) the curve is a left-handed helix as shown in Fig. 4-10(b). Since the sign of τ is an intrinsic property, we conclude that these two curves cannot be superimposed.

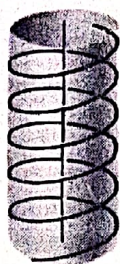
(a) Right-handed helix, $\tau > 0$ (b) Left-handed helix, $\tau < 0$

Fig. 4-10

If the torsion vanishes identically along a curve $\mathbf{x} = \mathbf{x}(s)$, i.e. if $\tau \equiv 0$, then $\dot{\mathbf{b}} = -\tau \mathbf{n} \equiv 0$. Hence $\mathbf{b} = \text{constant} \equiv \mathbf{b}_0$. Now consider

$$\frac{d}{ds}(\mathbf{x} \cdot \mathbf{b}_0) = \dot{\mathbf{x}} \cdot \mathbf{b}_0 = \mathbf{t} \cdot \mathbf{b}_0$$

Since $\mathbf{t}$ and $\mathbf{b}_0$ are orthogonal, then $\frac{d}{ds}(\mathbf{x} \cdot \mathbf{b}_0) \equiv 0$ and, integrating again,

$$\mathbf{x} \cdot \mathbf{b}_0 = \text{constant} \quad (4.16)$$

That is, $\mathbf{x} = \mathbf{x}(s)$ is a *plane curve* confined to the plane $\mathbf{x} \cdot \mathbf{b}_0 = \text{constant}$. In particular, $\mathbf{x} = \mathbf{x}(s)$ lies in its osculating plane as shown in Fig. 4-11. The converse is also true. Thus

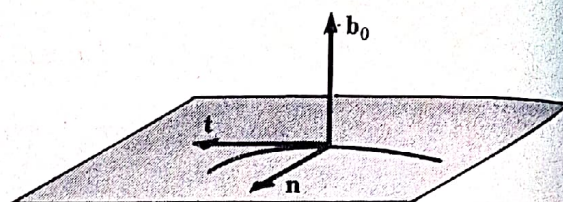


Fig. 4-11

Theorem 4.4. A curve of class ≥ 3 along which $\mathbf{n}$ is of class C^1 is a plane curve if and only if its torsion vanishes identically.

Unless otherwise stated we shall assume that our curves are **regular curves** of class ≥ 3 and that along the curve $\mathbf{n}$ is of class C^1 . In this case it follows that τ is continuous and that $\kappa, \mathbf{t}, \mathbf{n}$ and $\mathbf{b}$ are of class C^1 .

A convenient formula is also available for torsion in terms of an arbitrary representation. In Problem 4.19, page 77, we prove

Theorem 4.5. At a point on a curve $\mathbf{x} = \mathbf{x}(t)$ at which $\kappa \neq 0$,

$$\tau = \frac{[\mathbf{x}' \mathbf{x}'' \mathbf{x}''']}{|\mathbf{x}' \times \mathbf{x}''|^2}$$

SPHERICAL INDICATRICES

The unit tangent vectors along a curve C generate a curve Γ on the sphere of radius 1 about the origin as shown in Fig. 4-12. The curve Γ is called the *spherical indicatrix* of $\mathbf{t}$.

If $\mathbf{x} = \mathbf{x}(s)$ is a natural representation of C , then $\mathbf{x}_1 = \mathbf{t}(s) = \dot{\mathbf{x}}(s)$ will be a representation of Γ . However, in general s will not be a natural parameter along $\mathbf{x}_1 = \mathbf{t}(s)$, since

$$\left| \frac{d\mathbf{x}_1}{ds} \right| = \left| \frac{d\mathbf{t}}{ds} \right| = |\dot{\mathbf{t}}| = |\kappa|$$

In fact $\mathbf{x}_1 = \mathbf{t}(s)$ is a natural representation of Γ iff the curvature $|\kappa| \equiv 1$ along $\mathbf{x} = \mathbf{x}(s)$.

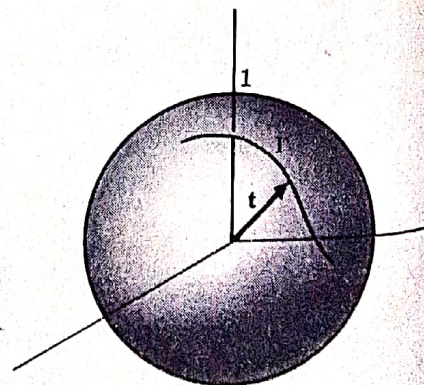


Fig. 4-12

Similarly one considers the spherical indicatrix of the unit normal $\mathbf{x}_2 = \mathbf{n}(s)$ and the spherical indicatrix $\mathbf{x}_3 = \mathbf{b}(s)$ of the unit binormal.

Example 4.10.

Referring to the helix in Examples 4.8 and 4.9, we have

$$\mathbf{x} = a(\cos t)\mathbf{e}_1 + a(\sin t)\mathbf{e}_2 + bte_3 \quad a > 0, b \neq 0$$

$$\mathbf{t} = (a^2 + b^2)^{-1/2}(-a(\sin t)\mathbf{e}_1 + a(\cos t)\mathbf{e}_2 + b\mathbf{e}_3)$$

$$\mathbf{n} = -((\cos t)\mathbf{e}_1 + (\sin t)\mathbf{e}_2)$$

$$\mathbf{b} = (a^2 + b^2)^{-1/2}(b(\sin t)\mathbf{e}_1 - b(\cos t)\mathbf{e}_2 + a\mathbf{e}_3)$$

Observe that $\mathbf{t}, \mathbf{n}, \mathbf{b}$ all have constant components with respect to $\mathbf{e}_3$, so that their spherical images are circles about the x_3 axis.

The radius of the spherical indicatrix of $\mathbf{t}$ and hence its radius of curvature, is

$$\rho_t = \left(1 - \frac{b^2}{a^2 + b^2}\right)^{1/2} = \frac{a}{(a^2 + b^2)^{1/2}}$$

The radii of curvature of the spherical indicatrix of $\mathbf{n}$ and $\mathbf{b}$ are $\rho_n = 1$ and $\rho_b = \frac{b}{(a^2 + b^2)^{1/2}}$ respectively.

Solved Problems

TANGENT LINE AND NORMAL PLANE

4.1. Find the equations of the tangent line and normal plane to the curve

$$\mathbf{x} = (1+t)\mathbf{e}_1 - t^2\mathbf{e}_2 + (1+t^3)\mathbf{e}_3$$

at $t = 1$.

$$\mathbf{x}' = \mathbf{e}_1 - 2t\mathbf{e}_2 + 3t^2\mathbf{e}_3, \quad \mathbf{x}(1) = 2\mathbf{e}_1 - \mathbf{e}_2 + 2\mathbf{e}_3, \quad \mathbf{x}'(1) = \mathbf{e}_1 - 2\mathbf{e}_2 + 3\mathbf{e}_3.$$

The equation of the tangent line at $t = 1$ is

$$\mathbf{y} = \mathbf{x}(1) + k\mathbf{x}'(1) \quad \text{or} \quad \mathbf{y} = (2+k)\mathbf{e}_1 - (1+2k)\mathbf{e}_2 + (2+3k)\mathbf{e}_3$$

The equation of the normal plane is

$$(\mathbf{y} - \mathbf{x}(1)) \cdot \mathbf{x}'(1) = 0 \quad \text{or} \quad (y_1 - 2) + (y_2 + 1)(-2) + (y_3 - 2)3 = 0 \quad \text{or} \quad y_1 - 2y_2 + 3y_3 = 10$$

4.2. Find the intersection of the x_1x_2 plane and the tangent lines to the helix

$$\mathbf{x} = (\cos t)\mathbf{e}_1 + (\sin t)\mathbf{e}_2 + t\mathbf{e}_3 \quad (t > 0)$$

The tangent line at an arbitrary point $\mathbf{x}$ is

$$\mathbf{y} = \mathbf{x} + k\mathbf{x}' \quad \text{or} \quad \mathbf{y} = (\cos t - k \sin t)\mathbf{e}_1 + (\sin t + k \cos t)\mathbf{e}_2 + (t+k)\mathbf{e}_3$$

or, using $\mathbf{x}$ as the position vector,

$$x_1 = \cos t - k \sin t, \quad x_2 = \sin t + k \cos t, \quad x_3 = t + k$$

The equation of the x_1x_2 plane is $x_3 = 0$. Hence along the intersection, $t + k = 0$ or $k = -t$. Thus the intersection is the curve

$$x_1 = \cos t + t \sin t, \quad x_2 = \sin t - t \cos t, \quad x_3 = 0$$

- 4.3. Show that the tangent vectors along the curve $\mathbf{x} = ate_1 + bt^2e_2 + t^3e_3$ where $2b^2$ make a constant angle with the vector $\mathbf{a} = e_1 + e_3$.

$$\mathbf{x}' = ae_1 + 2bte_2 + 3t^2e_3$$

$$|\mathbf{x}'| = (a^2 + 4b^2t^2 + 9t^4)^{1/2} = (a^2 + 6at^2 + 9t^4)^{1/2} = a + 3t^2$$

where we have used $2b^2 = 3a$. Then the angle between the tangent $\mathbf{x}'$ and $\mathbf{a}$ is

$$\cos^{-1} \left\{ \frac{(\mathbf{x}' \cdot \mathbf{a})}{|\mathbf{x}'| |\mathbf{a}|} \right\} = \cos^{-1} \left\{ \frac{a + 3t^2}{(a + 3t^2)\sqrt{2}} \right\} = \cos^{-1} (1/\sqrt{2}) = \pi/4$$

- 4.4. A curve is called a *general* or *cylindrical helix* or a curve of *constant slope*, if, as in preceding problem, there is a fixed vector in space, called the *axis* of the helix, such that the angle α between the tangent vectors and the axis is constant. We exclude case $\alpha = 0$ for which the tangent vectors are all parallel, in which case (see Problem 4.28) the curve is a straight line. Show that a general helix has a natural representation of the form

$$\mathbf{x} = x_1(s^*)e_1 + x_2(s^*)e_2 + s^*(\cos \alpha)e_3$$

We suppose that the helix is positioned so that the origin is on the curve and e_3 is parallel to the axis of the helix. Then

$$\cos \alpha = \cos \angle(\mathbf{x}', e_3) = \mathbf{x}' \cdot e_3 = \dot{x}_3 = \dot{x}_3$$

Integrating $\dot{x}_3 = \cos \alpha$ gives

$$x_3 = s \cos \alpha + c, \quad c = \text{constant}$$

If $\alpha \neq \pi/2$, set $s^* = s + c/(\cos \alpha)$. Then $x_3 = s^* \cos \alpha$ and we have a natural representation of the form

$$\mathbf{x} = x_1(s^*)e_1 + x_2(s^*)e_2 + s^*(\cos \alpha)e_3$$

as required. If $\alpha = \pi/2$, then $x_3 = c = 0$ since the origin is on the curve. Thus

$$\mathbf{x} = x_1(s)e_1 + x_2(s)e_2$$

In this case the curve is confined to the x_1x_2 plane.

CURVATURE

- 4.5. Find the curvature vector $\mathbf{k}$ and curvature $|\kappa|$ on the curve

$$\mathbf{x} = te_1 + \frac{1}{2}t^2e_2 + \frac{1}{3}t^3e_3$$

at the point $t = 1$.

$$\mathbf{x}' = e_1 + te_2 + t^2e_3, \quad |\mathbf{x}'| = (1 + t^2 + t^4)^{1/2}$$

$$\mathbf{t} = \mathbf{x}'/|\mathbf{x}'| = (1 + t^2 + t^4)^{-1/2}(e_1 + te_2 + t^2e_3)$$

$$\begin{aligned} \mathbf{t}' &= (1 + t^2 + t^4)^{-1/2}(e_2 + 2te_3) - (e_1 + te_2 + t^2e_3)(1 + t^2 + t^4)^{-3/2}(t + 2t^3) \\ &= -(1 + t^2 + t^4)^{-3/2}[(2t^3 + t)e_1 + (t^4 - 1)e_2 - (t^3 + 2t)e_3] \end{aligned}$$

$$\mathbf{k} = \dot{\mathbf{t}} = \mathbf{t}'/|\mathbf{x}'| = -(1 + t^2 + t^4)^{-2}[(2t^3 + t)e_1 + (t^4 - 1)e_2 - (t^3 + 2t)e_3]$$

At $t = 1$ we have $\mathbf{k} = -\frac{1}{8}(e_1 - e_3)$ and $|\kappa| = |\mathbf{k}| = \frac{1}{8}\sqrt{2}$.

- 4.6. Show that the curvature $|\kappa^*|$ of the projection of a general helix (see Problem 4.4) onto a plane normal to its axis is given by $|\kappa^*| = |\kappa|/\sin^2 \alpha$ where $\alpha \neq 0$ is the angle between the axis and the tangent vectors of the helix and $|\kappa|$ is the curvature of the helix.

Let $\mathbf{x} = \mathbf{x}(s)$ be the helix and $\mathbf{u}$, a vector of unit length, its axis. As indicated in Fig. 4-13, the projection on a plane perpendicular to $\mathbf{u}$ and passing through the origin is the curve

$$\mathbf{x}^* = \mathbf{x}(s) - (\mathbf{u} \cdot \mathbf{x}(s))\mathbf{u}$$

Note that in general s will not be a natural parameter for the projection $\mathbf{x}^* = \mathbf{x}^*(s)$. Differentiating gives

$$\frac{d\mathbf{x}^*}{ds} = \mathbf{t} - (\mathbf{u} \cdot \mathbf{t})\mathbf{u} = \mathbf{t} - (\cos \alpha)\mathbf{u}$$

and

$$\begin{aligned} \left| \frac{d\mathbf{x}^*}{ds} \right| &= \left[\frac{d\mathbf{x}^*}{ds} \cdot \frac{d\mathbf{x}^*}{ds} \right]^{1/2} \\ &= [(\mathbf{t} \cdot \mathbf{t}) - 2(\cos \alpha)(\mathbf{t} \cdot \mathbf{u}) + (\cos^2 \alpha)(\mathbf{u} \cdot \mathbf{u})]^{1/2} \\ &= [1 - 2\cos^2 \alpha + \cos^2 \alpha]^{1/2} = \sin \alpha, \end{aligned}$$

$$0 < \alpha < \pi$$

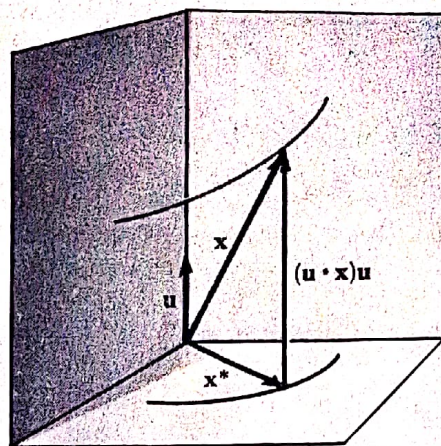


Fig. 4-13

Thus $\mathbf{t}^* = \frac{d\mathbf{x}^*}{ds} / \left| \frac{d\mathbf{x}^*}{ds} \right| = \frac{\mathbf{t} - (\cos \alpha)\mathbf{u}}{\sin \alpha}$ and $\frac{dt^*}{ds} = \dot{\mathbf{t}} / \sin \alpha$

and finally $|\kappa^*| = |\dot{\mathbf{t}}^*| = \left| \frac{dt^*}{ds} \right| / \left| \frac{d\mathbf{x}^*}{ds} \right| = |\dot{\mathbf{t}}| / \sin^2 \alpha = |\kappa| / \sin^2 \alpha$

4.7. Prove Theorem 4.2: Along a curve $\mathbf{x} = \mathbf{x}(t)$ of class ≥ 2 , $|\kappa| = |\mathbf{x}' \times \mathbf{x}''| / |\mathbf{x}'|^3$.

$$\mathbf{x}' = \frac{d\mathbf{x}}{dt} = \frac{d\mathbf{x}}{ds} \frac{ds}{dt} = \dot{\mathbf{x}} s', \quad \mathbf{x}'' = \frac{d}{dt}(\dot{\mathbf{x}} s') = \dot{\mathbf{x}} \frac{ds'}{dt} + s' \frac{d\dot{\mathbf{x}}}{dt} = \dot{\mathbf{x}} s'' + (s')^2 \ddot{\mathbf{x}}$$

$$\mathbf{x}' \times \mathbf{x}'' = (\dot{\mathbf{x}} s') \times (\dot{\mathbf{x}} s'' + \ddot{\mathbf{x}} (s')^2) = (s')^3 (\dot{\mathbf{x}} \times \ddot{\mathbf{x}}) = |\mathbf{x}'|^3 (\dot{\mathbf{x}} \times \ddot{\mathbf{x}})$$

where we have used $s' = ds/dt = |\mathbf{x}'|$. Then

$$|\mathbf{x}' \times \mathbf{x}''| = |\mathbf{x}'|^3 |\dot{\mathbf{x}} \times \ddot{\mathbf{x}}| = |\mathbf{x}'|^3 |\dot{\mathbf{x}}| |\ddot{\mathbf{x}}| \sin \angle(\dot{\mathbf{x}}, \ddot{\mathbf{x}})$$

But $\dot{\mathbf{x}} = \mathbf{t}$ and $\ddot{\mathbf{x}} = \dot{\mathbf{t}}$ are orthogonal, $|\dot{\mathbf{x}}| = 1$, and $|\ddot{\mathbf{x}}| = |\dot{\mathbf{t}}| = |\kappa|$. Hence $|\kappa| = |\mathbf{x}' \times \mathbf{x}''| / |\mathbf{x}'|^3$.

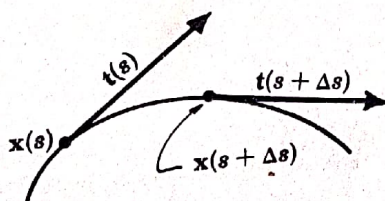
4.8. Show that a curve $\mathbf{x} = \mathbf{x}(t)$ of class ≥ 2 is a straight line if $\mathbf{x}'(t)$ and $\mathbf{x}''(t)$ are linearly dependent for all t .

Since $|\mathbf{x}' \times \mathbf{x}''| = 0$ if $\mathbf{x}'$ and $\mathbf{x}''$ are dependent, we have $|\kappa| = |\mathbf{x}' \times \mathbf{x}''| / |\mathbf{x}'|^3 = 0$ for all t . Hence from Theorem 4.1, page 64, $\mathbf{x} = \mathbf{x}(t)$ is a straight line.

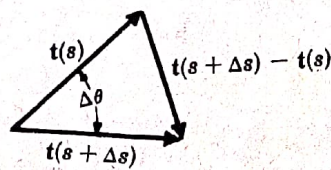
4.9. Let $\mathbf{x} = \mathbf{x}(s)$ be of class ≥ 2 and let $\Delta\theta$ denote the angle between the unit tangent $\mathbf{t}(s)$ at $\mathbf{x}(s)$ and $\mathbf{t}(s + \Delta s)$ at a neighboring point $\mathbf{x}(s + \Delta s)$, $\Delta s > 0$, as shown in Fig. 4-14(a). Show that the curvature

$$|\kappa| = \lim_{\Delta s \rightarrow 0} \frac{\Delta\theta}{\Delta s} = \frac{d\theta}{ds}$$

Namely, $|\kappa|$ is a measure of rate of change of direction of the tangent with respect to arc length.



(a)



(b)

Fig. 4-14

Since $\mathbf{t}$ is a unit vector, $|\mathbf{t}(s + \Delta s) - \mathbf{t}(s)|$ is the base of an isosceles triangle with sides length 1, as shown in Fig. 4-14(b). Hence

$$|\mathbf{t}(s + \Delta s) - \mathbf{t}(s)| = 2 \sin\left(\frac{1}{2}\Delta\theta\right) = \Delta\theta + o(\Delta\theta)$$

where we have used the Taylor expansion for the sine function. Then

$$\begin{aligned} |\kappa| &= \left| \frac{d\mathbf{t}}{ds} \right| = \left| \lim_{\Delta s \rightarrow 0} \frac{\mathbf{t}(s + \Delta s) - \mathbf{t}(s)}{\Delta s} \right| = \lim_{\Delta s \rightarrow 0} \left| \frac{\mathbf{t}(s + \Delta s) - \mathbf{t}(s)}{\Delta s} \right| \\ &= \lim_{\Delta s \rightarrow 0} \frac{\Delta\theta + o(\Delta\theta)}{\Delta s} = \lim_{\Delta s \rightarrow 0} \left[\frac{\Delta\theta}{\Delta s} \left(1 + \frac{o(\Delta\theta)}{\Delta\theta} \right) \right] \end{aligned}$$

Since $\lim_{\Delta s \rightarrow 0} \Delta\theta = 0$, then $\lim_{\Delta s \rightarrow 0} \frac{o(\Delta\theta)}{\Delta\theta} = 0$ and $|\kappa| = \lim_{\Delta s \rightarrow 0} \frac{\Delta\theta}{\Delta s} = \frac{d\theta}{ds}$.

4.10. Show that a curve $\mathbf{x} = \mathbf{x}(s)$ of class ≥ 2 is a straight line if all tangent lines have common intersection.

Suppose $\mathbf{y}_0$ is a common intersection of the tangent lines $\mathbf{y} = \mathbf{x}(s) + k\mathbf{t}(s)$. Then for each along the curve, there exists $k = k(s)$ such that

$$\mathbf{y}_0 = \mathbf{x}(s) + k(s)\mathbf{t}(s)$$

Note that along a tangent line $\mathbf{y} = \mathbf{x}(s) + k\mathbf{t}(s)$ we have $k = 0$ if and only if $\mathbf{y}$ is on the curve. Thus if we assume that the curve is simple, which is the case locally, $k(s) \neq 0$ except perhaps for a single value of s , say s_0 . Differentiating the above,

$$0 = \dot{\mathbf{x}} + \dot{k}\mathbf{t} + k\dot{\mathbf{t}} = (1 + \dot{k})\mathbf{t} + k\dot{\mathbf{t}}$$

Multiplying by $\dot{\mathbf{t}}$,

$$0 = (1 + \dot{k})(\mathbf{t} \cdot \dot{\mathbf{t}}) + k(\dot{\mathbf{t}} \cdot \dot{\mathbf{t}})$$

Since $\dot{\mathbf{t}}$ is orthogonal to $\mathbf{t}$, then $0 = k|\dot{\mathbf{t}}|^2$ or $0 = k|\kappa|^2$. Since $k(s) \neq 0$ for $s \neq s_0$, $|\kappa| = 0$ for $s \neq s_0$. But $|\kappa|$ is continuous. Hence $|\kappa| = 0$ for all s , and it follows from Theorem 4.1, page that $\mathbf{x} = \mathbf{x}(s)$ is a straight line.

MOVING TRIHEDRON

4.11. Find a continuous unit principal normal and unit binormal along the curve

$$\mathbf{x} = (3t - t^3)\mathbf{e}_1 + 3t^2\mathbf{e}_2 + (3t + t^3)\mathbf{e}_3$$

$$\mathbf{x}' = (3 - 3t^2)\mathbf{e}_1 + 6t\mathbf{e}_2 + (3 + 3t^2)\mathbf{e}_3$$

$$|\mathbf{x}'| = 3[(1 - t^2)^2 + (2t)^2 + (1 + t^2)^2]^{1/2} = 3\sqrt{2}(1 + t^2 + t^4)^{1/2} = 3\sqrt{2}(1 + t^2)$$

$$\mathbf{t} = \frac{\mathbf{x}'}{|\mathbf{x}'|} = \frac{1}{\sqrt{2}(1 + t^2)} [(1 - t^2)\mathbf{e}_1 + 2t\mathbf{e}_2 + (1 + t^2)\mathbf{e}_3]$$

$$\mathbf{k} = \dot{\mathbf{t}} = \frac{\mathbf{t}'}{|\mathbf{x}'|} = \frac{-2t\mathbf{e}_1 + (1 - t^2)\mathbf{e}_2}{3(1 + t^2)^3}, \quad |\mathbf{k}| = \frac{[(2t)^2 + (1 - t^2)^2]^{1/2}}{3(1 + t^2)^3} = \frac{1}{3(1 + t^2)^2}$$

Since $\mathbf{k} \neq 0$ for all t , we can choose

$$\mathbf{n} = \frac{\mathbf{k}}{|\mathbf{k}|} = \frac{-2t}{1 + t^2}\mathbf{e}_1 + \frac{1 - t^2}{1 + t^2}\mathbf{e}_2$$

and

$$\begin{aligned} \mathbf{b} &= \mathbf{t} \times \mathbf{n} = \frac{1}{\sqrt{2}(1 + t^2)^2} \det \begin{pmatrix} \mathbf{e}_1 & 1 - t^2 & -2t \\ \mathbf{e}_2 & 2t & 1 - t^2 \\ \mathbf{e}_3 & 1 + t^2 & 0 \end{pmatrix} \\ &= \frac{1}{\sqrt{2}(1 + t^2)^2} [-(1 + t^2)(1 - t^2)\mathbf{e}_1 - 2t(1 + t^2)\mathbf{e}_2 + (1 + t^2)^2\mathbf{e}_3] \\ &= \frac{1}{\sqrt{2}(1 + t^2)} [(t^2 - 1)\mathbf{e}_1 - 2t\mathbf{e}_2 + (1 + t^2)\mathbf{e}_3] \end{aligned}$$

- 4.12. Show that along a curve $\mathbf{x} = \mathbf{x}(t)$ the vector $\mathbf{x}''$ is parallel to the osculating plane and that its components with respect to $\mathbf{t}$ and $\mathbf{n}$ are $|\mathbf{x}'|'$ and $\kappa|\mathbf{x}'|^2$ respectively.

Differentiating $\mathbf{x}' = \frac{d\mathbf{x}}{ds} \frac{ds}{dt} = \mathbf{t}s'$ with respect to t , we obtain

$$\mathbf{x}'' = \mathbf{t}s'' + \dot{\mathbf{t}}s' = \mathbf{t}s'' + \dot{\mathbf{t}}s'^2 = \mathbf{t}s'' + \kappa s'^2 \mathbf{n}$$

where from equation (4.6), $\dot{\mathbf{t}} = \kappa \mathbf{n}$. It follows that $\mathbf{x}''$ is parallel to the osculating plane and that its components with respect to $\mathbf{t}$ and $\mathbf{n}$ are $|\mathbf{x}'|' = s''$ and $\kappa|\mathbf{x}'|^2 = \kappa s'^2$ respectively.

- 4.13. (a) If $\mathbf{x}'$ and $\mathbf{x}''$ are linearly independent at a point $\mathbf{x}$ along $\mathbf{x} = \mathbf{x}(t)$, show that the osculating plane at $\mathbf{x}$ is $[(\mathbf{y} - \mathbf{x})\mathbf{x}'\mathbf{x}''] = 0$. (b) Use this formula to find the osculating plane to the curve $\mathbf{x} = t\mathbf{e}_1 + t^2\mathbf{e}_2 + t^3\mathbf{e}_3$ at $t = 1$.

(a) We saw in the above problem that $\mathbf{x}''$ is parallel to the osculating plane and we know that $\mathbf{x}'$, being a multiple of $\mathbf{t}$, is parallel to the osculating plane. Since we are given that $\mathbf{x}'$ and $\mathbf{x}''$ are independent, it follows that $\mathbf{x}' \times \mathbf{x}''$ is a nonzero vector normal to the osculating plane at $\mathbf{x}$. Therefore the equation of the osculating plane at $\mathbf{x}$ is $[(\mathbf{y} - \mathbf{x})\mathbf{x}'\mathbf{x}''] = 0$.

(b) $\mathbf{x}' = \mathbf{e}_1 + 2t\mathbf{e}_2 + 3t^2\mathbf{e}_3$, $\mathbf{x}'' = 2\mathbf{e}_2 + 6t\mathbf{e}_3$

Thus the osculating plane at $t = 1$ is $[(\mathbf{y} - \mathbf{x}(1))\mathbf{x}'(1)\mathbf{x}''(1)] = 0$ or

$$\det \begin{pmatrix} y_1 - 1 & 1 & 0 \\ y_2 - 1 & 2 & 2 \\ y_3 - 1 & 3 & 6 \end{pmatrix} = 0$$

from which

$$(y_1 - 1)6 - (y_2 - 1)6 + (y_3 - 1)2 = 0 \quad \text{or} \quad 3y_1 - 3y_2 + y_3 = 1$$

- 4.14. Show that the points on the helix $\mathbf{x} = a(\cos t)\mathbf{e}_1 + a(\sin t)\mathbf{e}_2 + bte_3$ at which the osculating planes pass through a fixed point are confined to a plane.

Let $\mathbf{x}_\lambda = a(\cos t_\lambda)\mathbf{e}_1 + a(\sin t_\lambda)\mathbf{e}_2 + bt_\lambda\mathbf{e}_3$ denote the points on the helix at which the osculating plane passes through the point $\mathbf{y}_0 = y_{01}\mathbf{e}_1 + y_{02}\mathbf{e}_2 + y_{03}\mathbf{e}_3$. It is easily verified that $\mathbf{x}'$ and $\mathbf{x}''$ are independent for all t . Thus the equation of the osculating plane at $\mathbf{x}$ is $[(\mathbf{y} - \mathbf{x})\mathbf{x}'\mathbf{x}''] = 0$. Hence for each λ ,

$$[(\mathbf{y}_0 - \mathbf{x}(t_\lambda))\mathbf{x}'(t_\lambda)\mathbf{x}''(t_\lambda)] = 0$$

or

$$\det \begin{pmatrix} y_{01} - a \cos t_\lambda & -a \sin t_\lambda & -a \cos t_\lambda \\ y_{02} - a \sin t_\lambda & a \cos t_\lambda & -a \sin t_\lambda \\ y_{03} - bt_\lambda & b & 0 \end{pmatrix} = 0$$

or, expanding,

$$(by_{02})(a \cos t_\lambda) - (by_{01})(a \sin t_\lambda) + (a^2)(bt_\lambda) = a^2y_{03}$$

Thus the $\mathbf{x}_\lambda = a(\cos t_\lambda)\mathbf{e}_1 + a(\sin t_\lambda)\mathbf{e}_2 + bt_\lambda\mathbf{e}_3$ lie on the plane $by_{02}x_1 - by_{01}x_2 + a^2x_3 = a^2y_{03}$.

- 4.15. Prove Theorem 4.3: If $\mathbf{x}(s_0)$ is a point of inflection on an analytic curve $\mathbf{x} = \mathbf{x}(s)$ which is not a straight line, then there exists a continuous principal normal unit vector $\mathbf{n}(s)$ to the curve in a neighborhood of s_0 .

Let $\mathbf{x}^{(k)}(s_0)$ be the first nonvanishing derivative of $\mathbf{x}(s)$ at s_0 of order $k > 1$. Since $\mathbf{x} = \mathbf{x}(s)$ is not a straight line, such a nonvanishing derivative exists. In fact $k > 2$, since $\dot{\mathbf{i}}(s_0) = \ddot{\mathbf{x}}(s_0) = 0$ at a point of inflection. Thus we can write

$$\mathbf{x} = \mathbf{x}(s_0) + \dot{\mathbf{x}}(s_0)(s - s_0) + \frac{\mathbf{x}^{(k)}(s_0)}{k!}(s - s_0)^k + \frac{\mathbf{x}^{(k+1)}(s_0)(s - s_0)^{k+1}}{(k+1)!} + \dots$$

Then
$$\mathbf{t} = \dot{\mathbf{x}} = \dot{\mathbf{x}}(s_0) + \frac{\mathbf{x}^{(k)}(s_0)(s-s_0)^{k-1}}{(k-1)!} + \frac{\mathbf{x}^{(k+1)}(s_0)(s-s_0)^k}{k!} + \dots$$

and
$$\begin{aligned} \mathbf{k} = \dot{\mathbf{t}} &= \frac{\mathbf{x}^{(k)}(s_0)(s-s_0)^{k-2}}{(k-2)!} + \frac{\mathbf{x}^{(k+1)}(s_0)(s-s_0)^{k-1}}{(k-1)!} + \dots \\ &= (s-s_0)^{k-2} \left[\frac{\mathbf{x}^{(k)}(s_0)}{(k-2)!} + \frac{\mathbf{x}^{(k+1)}(s_0)(s-s_0)}{(k-1)!} + \dots \right] = (s-s_0)^{k-2} \mathbf{w}(s) \end{aligned}$$

where $\mathbf{w}(s)$ is analytic at s_0 and $\mathbf{w}(s_0) = \frac{\mathbf{x}^{(k)}(s_0)}{(k-2)!} \neq 0$. Since $\mathbf{w}(s)$ is certainly continuous, there exists a neighborhood $S_\delta(s_0)$ such that $\mathbf{w}(s) \neq 0$ for s in $S_\delta(s_0)$. Now consider the vector $\mathbf{n} = \mathbf{w}(s)/|\mathbf{w}(s)|$. For s in $S_\delta(s_0)$, the vector $\mathbf{n}$ is continuous, of unit length, and $\mathbf{k} = \dot{\mathbf{t}}$ is a multiple of $\mathbf{n}$; for,

$$\mathbf{k} = (s-s_0)^{k-2} \mathbf{w}(s) = (s-s_0)^{k-2} |\mathbf{w}(s)| \frac{\mathbf{w}(s)}{|\mathbf{w}(s)|} = (s-s_0)^{k-2} |\mathbf{w}(s)| \mathbf{n} = \kappa(s) \mathbf{n},$$

which is the required result. Note that for this $\mathbf{n}$, $\kappa = |\mathbf{k}|$ iff k is even $((s-s_0)^{k-2})$. Otherwise

$$\kappa = \begin{cases} |\mathbf{k}| & \text{for } s \geq s_0 \\ -|\mathbf{k}| & \text{for } s < s_0 \end{cases}$$

TORSION

4.16. Using the formulas of Theorems 4.2 and 4.5, find the curvature and torsion at the curve

$$\mathbf{x} = (3t - t^3)\mathbf{e}_1 + 3t^2\mathbf{e}_2 + (3t + t^3)\mathbf{e}_3$$

$$\begin{aligned} |\kappa| &= \frac{|\mathbf{x}' \times \mathbf{x}''|}{|\mathbf{x}'|^3} = \frac{|[(3-3t^2)\mathbf{e}_1 + 6t\mathbf{e}_2 + (3+3t^2)\mathbf{e}_3] \times [-6t\mathbf{e}_1 + 6\mathbf{e}_2 + 6t\mathbf{e}_3]|}{|(3-3t^2)\mathbf{e}_1 + 6t\mathbf{e}_2 + (3+3t^2)\mathbf{e}_3|^3} \\ &= \frac{18|(t^2-1)\mathbf{e}_1 - 2t\mathbf{e}_2 + (1+t^2)\mathbf{e}_3|}{27|(1-t^2)\mathbf{e}_1 + 2t\mathbf{e}_2 + (1+t^2)\mathbf{e}_3|^3} = \frac{2}{3(1+t^2)^2} \end{aligned}$$

$$\tau = \frac{[\mathbf{x}' \times \mathbf{x}'' \cdot \mathbf{x}''']}{|\mathbf{x}' \times \mathbf{x}''|^2} = \frac{(\mathbf{x}' \times \mathbf{x}'') \cdot \mathbf{x}'''}{|\mathbf{x}' \times \mathbf{x}''|^2} = \frac{18[(t^2-1)\mathbf{e}_1 - 2t\mathbf{e}_2 + (1+t^2)\mathbf{e}_3] \cdot 6[-\mathbf{e}_1 + \mathbf{e}_3]}{18^2|(t^2-1)\mathbf{e}_1 - 2t\mathbf{e}_2 + (1+t^2)\mathbf{e}_3|^2} = \frac{1}{3(1+t^2)^2}$$

Observe that along this curve, $|\kappa| = \tau$.

4.17. Show that along a curve $\mathbf{x} = \mathbf{x}(s)$, $\ddot{\mathbf{x}} = -\kappa^2 \mathbf{t} + \dot{\kappa} \mathbf{n} + \tau \kappa \mathbf{b}$.

Differentiating $\ddot{\mathbf{x}} = \dot{\mathbf{t}} = \kappa \mathbf{n}$, we have

$$\ddot{\mathbf{x}} = \kappa \dot{\mathbf{n}} + \dot{\kappa} \mathbf{n} = \kappa \frac{d}{ds}(\mathbf{b} \times \mathbf{t}) + \dot{\kappa} \mathbf{n} = \kappa(\mathbf{b} \times \dot{\mathbf{t}} + \dot{\mathbf{b}} \times \mathbf{t}) + \dot{\kappa} \mathbf{n}$$

where we use the fact that $\mathbf{t}, \mathbf{n}, \mathbf{b}$ is a right-handed orthonormal triplet and hence $\mathbf{n} = \mathbf{b} \times \mathbf{t}$. Again using $\dot{\mathbf{t}} = \kappa \mathbf{n}$ and also $\dot{\mathbf{b}} = -\tau \mathbf{n}$, we have

$$\ddot{\mathbf{x}} = \kappa^2(\mathbf{b} \times \mathbf{n}) - \kappa\tau(\mathbf{n} \times \mathbf{t}) + \dot{\kappa} \mathbf{n} = \kappa^2(-\mathbf{t}) + \kappa\tau \mathbf{b} + \dot{\kappa} \mathbf{n}$$

4.18. Show that along a curve $\mathbf{x} = \mathbf{x}(s)$, $[\dot{\mathbf{x}} \ddot{\mathbf{x}} \ddot{\mathbf{x}}] = \kappa^2 \tau$.

Using the result of the above problem, we have

$$\dot{\mathbf{x}} \times \ddot{\mathbf{x}} = \mathbf{t} \times \ddot{\mathbf{x}} = \kappa \mathbf{n} \times (-\kappa^2 \mathbf{t} + \dot{\kappa} \mathbf{n} + \tau \kappa \mathbf{b}) = -\kappa^3(\mathbf{n} \times \mathbf{t}) + \kappa \dot{\kappa}(\mathbf{n} \times \mathbf{n}) + \tau \kappa^2(\mathbf{n} \times \mathbf{b}) = \kappa^3 \mathbf{b} + \tau \kappa^2 \mathbf{t}$$

since $\mathbf{t}, \mathbf{n}, \mathbf{b}$ is a right-handed orthonormal triplet. Finally,

$$[\dot{\mathbf{x}} \ddot{\mathbf{x}} \ddot{\mathbf{x}}] = \dot{\mathbf{x}} \cdot \ddot{\mathbf{x}} \times \ddot{\mathbf{x}} = \mathbf{t} \cdot \ddot{\mathbf{x}} \times \ddot{\mathbf{x}} = \kappa^3(\mathbf{t} \cdot \mathbf{b}) + \tau \kappa^2(\mathbf{t} \cdot \mathbf{t}) = \tau \kappa^2$$

since $\mathbf{t} \cdot \mathbf{b} = 0$ and $\mathbf{t} \cdot \mathbf{t} = 1$.

- 4.19. Prove Theorem 4.5: At a point on a curve $\mathbf{x} = \mathbf{x}(t)$ where $\kappa \neq 0$, the torsion is given by

$$\tau = \frac{[\mathbf{x}'\mathbf{x}''\mathbf{x}''']}{|\mathbf{x}' \times \mathbf{x}''|^2}$$

$$\dot{\mathbf{x}} = \frac{d\mathbf{x}}{ds} = \frac{d\mathbf{x}}{dt} \frac{dt}{ds} = \mathbf{x}'\dot{t}, \quad \ddot{\mathbf{x}} = \frac{d}{ds}(\mathbf{x}'\dot{t}) = \mathbf{x}'\ddot{t} + \left(\frac{d}{ds}\mathbf{x}'\right)\dot{t} = \mathbf{x}'\ddot{t} + \mathbf{x}''\dot{t}^2$$

$$\ddot{\mathbf{x}} = \frac{d}{ds}(\mathbf{x}'\dot{t} + \mathbf{x}''\dot{t}^2) = \mathbf{x}'\ddot{t} + \mathbf{x}''\dot{t}\ddot{t} + \mathbf{x}''2\dot{t}\ddot{t} + \mathbf{x}'''\dot{t}^3 = \mathbf{x}'\ddot{t} + 3\mathbf{x}''\dot{t}\ddot{t} + \mathbf{x}'''\dot{t}^3$$

Hence

$$\begin{aligned} [\ddot{\mathbf{x}}\ddot{\mathbf{x}}\ddot{\mathbf{x}}] &= (\mathbf{x}'\ddot{t}) \cdot (\mathbf{x}'\ddot{t} + \mathbf{x}''\dot{t}^2) \times (\mathbf{x}'\ddot{t} + 3\mathbf{x}''\dot{t}\ddot{t} + \mathbf{x}'''\dot{t}^3) \\ &= (\mathbf{x}'\ddot{t}) \cdot [3\dot{t}^2\ddot{t}(\mathbf{x}' \times \mathbf{x}'') + \ddot{t}\dot{t}^3(\mathbf{x}' \times \mathbf{x}''') + \dot{t}^2\ddot{t}(\mathbf{x}'' \times \mathbf{x}') + \dot{t}^5(\mathbf{x}'' \times \mathbf{x}''')] \\ &= 3\dot{t}^2\ddot{t}^2[\mathbf{x}' \times \mathbf{x}'] + \ddot{t}\dot{t}^4[\mathbf{x}' \times \mathbf{x}'''] + \dot{t}^3\ddot{t}[\mathbf{x}' \times \mathbf{x}'] + \dot{t}^6[\mathbf{x}' \times \mathbf{x}'''] \end{aligned}$$

Since $[\mathbf{x}' \times \mathbf{x}'] = 0$,

$$[\ddot{\mathbf{x}}\ddot{\mathbf{x}}\ddot{\mathbf{x}}] = \dot{t}^6[\mathbf{x}' \times \mathbf{x}'''] \quad \text{or} \quad [\ddot{\mathbf{x}}\ddot{\mathbf{x}}\ddot{\mathbf{x}}] = \frac{[\mathbf{x}' \times \mathbf{x}''']}{|\mathbf{x}'|^6}$$

since $\dot{t} = \frac{dt}{ds} = \frac{1}{ds/dt} = \frac{1}{|\mathbf{x}'|}$. Using the results of the above problem and $\kappa = \frac{|\mathbf{x}' \times \mathbf{x}''|}{|\mathbf{x}'|^3}$ (Theorem 4.2), we have

$$\tau = \frac{[\ddot{\mathbf{x}}\ddot{\mathbf{x}}\ddot{\mathbf{x}}]}{\kappa^2} = \frac{[\mathbf{x}' \times \mathbf{x}''']}{\kappa^2 |\mathbf{x}'|^6} = \frac{[\mathbf{x}' \times \mathbf{x}''']}{|\mathbf{x}' \times \mathbf{x}''|^2}$$

- 4.20. Show that a curve is a plane curve if all osculating planes have a common point of intersection.

The equation of the osculating plane at $\mathbf{x}$ is $(\mathbf{y} - \mathbf{x}) \cdot \mathbf{b} = 0$. Thus if $\mathbf{y}_0$ is the common intersection, $(\mathbf{y}_0 - \mathbf{x}) \cdot \mathbf{b} = 0$ for all s . Differentiating and using the fact that $\dot{\mathbf{x}} = \mathbf{t}$ and $\mathbf{b}$ are orthogonal, we have

$$(\mathbf{y}_0 - \mathbf{x}) \cdot \dot{\mathbf{b}} = 0 \quad \text{or} \quad \tau(\mathbf{y}_0 - \mathbf{x}) \cdot \mathbf{n} = 0$$

since $\dot{\mathbf{b}} = -\tau\mathbf{n}$. Now suppose the curve is not a plane curve. Then from Theorem 4.4, page 70, there exists a point s_0 and hence a neighborhood $S_\delta(s_0)$ where $\tau \neq 0$. Thus in $S_\delta(s_0)$, we have $(\mathbf{y}_0 - \mathbf{x}) \cdot \mathbf{n} = 0$; that is, $\mathbf{y}_0 - \mathbf{x}$ is orthogonal to $\mathbf{n}$. But also, $(\mathbf{y}_0 - \mathbf{x}) \cdot \mathbf{b} = 0$; hence $\mathbf{y}_0 - \mathbf{x}$ is parallel to $\mathbf{t}$. Thus there exists $k = k(s)$ such that for s in $S_\delta(s_0)$, $\mathbf{y}_0 - \mathbf{x} = k\mathbf{t}$ or $\mathbf{y}_0 = \mathbf{x} + k\mathbf{t}$; that is (see Problem 4.10), the point $\mathbf{y}_0$ is also the intersection of the tangent lines. But then in $S_\delta(s_0)$ the curve is a straight line, which is impossible since $\tau \neq 0$ in $S_\delta(s_0)$. Hence $\tau = 0$ for all s and the curve is a plane curve.

- 4.21. Show that a curve is a general helix if and only if τ/κ is constant where $\kappa \neq 0$ and $\tau = 0$ whenever $\kappa = 0$.

Suppose $\mathbf{x} = \mathbf{x}(s)$ is a general helix with axis $\mathbf{u}$ of unit length and that $\mathbf{t} \cdot \mathbf{u} = \cos \alpha$. Recall (Problem 4.4) that we have excluded the case $\alpha = 0$ which would correspond to a straight line parallel to $\mathbf{u}$. Thus for each s there is a unique unit vector $\mathbf{b}^*(s)$ orthogonal to $\mathbf{t}(s)$ such that $\mathbf{u} = \mathbf{t} \cos \alpha + \mathbf{b}^* \sin \alpha$. Evidently $\mathbf{b}^*(s)$ and also $\mathbf{n}^*(s) = \mathbf{b}^*(s) \times \mathbf{t}(s)$ are continuously differentiable. Differentiating $\mathbf{t} \cdot \mathbf{u} = \cos \alpha$ we obtain $\dot{\mathbf{t}} \cdot (\mathbf{t} \cos \alpha + \mathbf{b}^* \sin \alpha) = 0$ and, since $\dot{\mathbf{t}} \cdot \mathbf{t} = 0$, there follows $\dot{\mathbf{t}} \cdot \mathbf{b}^* = 0$. Thus $\mathbf{n} = \mathbf{n}^*$ and $\mathbf{b} = \mathbf{b}^*$ can be chosen respectively as the principal unit normal and unit binormal along the curve. From

$$0 = \dot{\mathbf{u}} = \dot{\mathbf{t}} \cos \alpha + \dot{\mathbf{b}} \sin \alpha = \kappa \mathbf{n} \cos \alpha - \tau \mathbf{n} \sin \alpha$$

we conclude that $\tau/\kappa = \cot \alpha = \text{constant}$ whenever $\kappa \neq 0$ and that $\tau = 0$ whenever $\kappa = 0$. Conversely, suppose that $\tau/\kappa = \cot \alpha$ whenever $\kappa \neq 0$ and $\tau = 0$ whenever $\kappa = 0$. Then $\kappa \cos \alpha - \tau \sin \alpha = 0$ and hence $\dot{\mathbf{t}} \cos \alpha + \dot{\mathbf{b}} \sin \alpha = 0$ or, integrating, $\mathbf{t} \cos \alpha + \mathbf{b} \sin \alpha = \mathbf{u} = \text{constant}$. Thus $\mathbf{u}$ is of unit length and $\mathbf{t} \cdot \mathbf{u} = \cos \alpha = \text{constant}$; that is, $\mathbf{x} = \mathbf{x}(s)$ is a helix.

- *4.22. Show that along a regular curve $\mathbf{x} = \mathbf{x}(s)$ of class ≥ 4 ,

$$[\ddot{\mathbf{x}} \ddot{\mathbf{x}} \mathbf{x}^{(4)}] = \kappa^5 \frac{d}{ds} \left(\frac{\tau}{\kappa} \right)$$

and hence $\mathbf{x} = \mathbf{x}(s)$ is a general helix if and only if $[\ddot{\mathbf{x}} \ddot{\mathbf{x}} \mathbf{x}^{(4)}] = 0$.

Referring to Problem 4.18, $\ddot{\mathbf{x}} \times \ddot{\mathbf{x}} = \kappa^3 \mathbf{b} + \tau \kappa^2 \mathbf{t}$. Differentiating,

$$\begin{aligned} \ddot{\mathbf{x}} \times \mathbf{x}^{(4)} &= \left(\frac{d}{ds} \kappa^3 \right) \mathbf{b} + \kappa^3 \dot{\mathbf{b}} + \left(\frac{d}{ds} \tau \kappa^2 \right) \mathbf{t} + \tau \kappa^2 \dot{\mathbf{t}} \\ &= \left(\frac{d}{ds} \kappa^3 \right) \mathbf{b} - \tau \kappa^3 \mathbf{n} + \left(\frac{d}{ds} \tau \kappa^2 \right) \mathbf{t} + \tau \kappa^3 \mathbf{n} = \left(\frac{d}{ds} \kappa^3 \right) \mathbf{b} + \left(\frac{d}{ds} \tau \kappa^2 \right) \mathbf{t} \end{aligned}$$

Then, applying Problem 4.17,

$$\begin{aligned} \ddot{\mathbf{x}} \cdot \ddot{\mathbf{x}} \times \mathbf{x}^{(4)} &= (-\kappa^2 \mathbf{t} + \kappa \mathbf{n} + \tau \kappa \mathbf{b}) \cdot \left[\left(\frac{d}{ds} \kappa^3 \right) \mathbf{b} + \left(\frac{d}{ds} \tau \kappa^2 \right) \mathbf{t} \right] = -\kappa^2 \frac{d}{ds} \tau \kappa^2 + \tau \kappa \frac{d}{ds} \kappa^3 \\ &= -\kappa^4 \dot{\tau} - 2\tau \kappa^3 \dot{\kappa} + 3\tau \kappa^3 \dot{\kappa} = -\kappa^3 [\kappa \dot{\tau} - \dot{\kappa} \tau] = -\kappa^5 \frac{d}{ds} \left(\frac{\tau}{\kappa} \right) \end{aligned}$$

Thus

$$[\ddot{\mathbf{x}} \ddot{\mathbf{x}} \mathbf{x}^{(4)}] = -[\ddot{\mathbf{x}} \ddot{\mathbf{x}} \mathbf{x}^{(4)}] = \kappa^5 \frac{d}{ds} \left(\frac{\tau}{\kappa} \right)$$

If $\mathbf{x} = \mathbf{x}(s)$ is a general helix, then everywhere on the curve either $\kappa = 0$ or τ/κ is constant that $[\ddot{\mathbf{x}} \ddot{\mathbf{x}} \mathbf{x}^{(4)}] = 0$. Conversely, if $[\ddot{\mathbf{x}} \ddot{\mathbf{x}} \mathbf{x}^{(4)}] = 0$ and if there are no points of inflection that $\kappa \neq 0$, then we can conclude that $\kappa/\tau = \text{constant}$ and therefore $\mathbf{x} = \mathbf{x}(s)$ is a general helix.

SPHERICAL INDICATRICES

- 4.23. Show that the tangent to the spherical indicatrix of the tangent to a curve C is parallel to the principal normal to C at corresponding points.

Let $\mathbf{x}_1 = \mathbf{t}(s)$ denote the spherical indicatrix of the tangent to the curve $\mathbf{x} = \mathbf{x}(s)$. The tangent vector to $\mathbf{x}_1 = \mathbf{t}(s)$ is

$$\frac{d\mathbf{x}_1}{ds} = \dot{\mathbf{t}}(s) = \kappa(s) \mathbf{n}(s)$$

which gives the required result.

- 4.24. Show that the curvature of the spherical indicatrix $\mathbf{x}_1 = \mathbf{t}(s)$ of the tangent to a curve C is $\kappa_1^2 = (\kappa^2 + \tau^2)/\kappa^2$.

From Theorem 4.2, page 64, $|\kappa_1| = \frac{|\dot{\mathbf{t}} \times \ddot{\mathbf{t}}|}{|\dot{\mathbf{t}}|^3} = \frac{|\ddot{\mathbf{x}} \times \ddot{\mathbf{x}}|}{|\kappa|^3}$. From Problem 4.18,

$$|\kappa_1| = \frac{|\kappa^3 \mathbf{b} + \tau \kappa^2 \mathbf{t}|}{|\kappa|^3} = \frac{|\kappa \mathbf{b} + \tau \mathbf{t}|}{|\kappa|}$$

or

$$\kappa_1^2 = \frac{|\kappa \mathbf{b} + \tau \mathbf{t}|^2}{\kappa^2} = \frac{(\kappa \mathbf{b} + \tau \mathbf{t}) \cdot (\kappa \mathbf{b} + \tau \mathbf{t})}{\kappa^2} = \frac{\kappa^2 + \tau^2}{\kappa^2}$$

- 4.25. Show that the torsion of the binormal indicatrix $\mathbf{x}_3 = \mathbf{b}(s)$ of a sufficiently regular curve is $\tau_3 = \frac{\tau \dot{\kappa} - \kappa \dot{\tau}}{\tau(\kappa^2 + \tau^2)}$.

We use Theorem 4.5, page 70, to find τ_3 . From $\dot{\mathbf{b}} = -\tau \mathbf{n}$ we have

$$\begin{aligned} -\ddot{\mathbf{b}} &= \dot{\tau} \mathbf{n} + \tau \dot{\mathbf{n}} = \dot{\tau} \mathbf{n} + \tau \frac{d}{ds} (\mathbf{b} \times \mathbf{t}) = \dot{\tau} \mathbf{n} + \tau [(\dot{\mathbf{b}} \times \mathbf{t}) + (\mathbf{b} \times \dot{\mathbf{t}})] \\ &= \dot{\tau} \mathbf{n} + \tau [-\tau (\mathbf{n} \times \mathbf{t}) + \kappa (\mathbf{b} \times \mathbf{n})] = \dot{\tau} \mathbf{n} + \tau^2 \mathbf{b} - \kappa \tau \mathbf{t} \end{aligned}$$

Then

$$\dot{\mathbf{b}} \times \ddot{\mathbf{b}} = (\tau \mathbf{n}) \times (\dot{\tau} \mathbf{n} + \tau^2 \mathbf{b} - \kappa \tau \mathbf{t}) = \tau^3 (\mathbf{n} \times \mathbf{b}) - \tau^2 \kappa (\mathbf{n} \times \mathbf{t}) = \tau^3 \mathbf{t} + \tau^2 \kappa \mathbf{b} = \tau^2 (\tau \mathbf{t} + \kappa \mathbf{b})$$

$$\begin{aligned}
 \text{and } \dot{\mathbf{b}} \times \ddot{\mathbf{b}} &= \frac{d}{ds}(\dot{\mathbf{b}} \times \ddot{\mathbf{b}}) = \frac{d}{ds}(\tau \dot{\mathbf{t}} + \kappa \mathbf{b}) = 2\tau \dot{\tau}(\dot{\mathbf{t}} + \kappa \mathbf{b}) + \tau^2(\dot{\mathbf{t}} + \tau \dot{\mathbf{t}} + \dot{\kappa} \mathbf{b} + \kappa \dot{\mathbf{b}}) \\
 &= 2\tau \dot{\tau}(\dot{\mathbf{t}} + \kappa \mathbf{b}) + \tau^2(\dot{\mathbf{t}} + \kappa \tau \mathbf{n} + \dot{\kappa} \mathbf{b} - \tau \kappa \mathbf{n}) \\
 &= 2\tau \dot{\tau}(\dot{\mathbf{t}} + \kappa \mathbf{b}) + \tau^2(\dot{\mathbf{t}} + \dot{\kappa} \mathbf{b}) = 3\tau^2 \dot{\mathbf{t}} + (2\tau \dot{\tau} \kappa + \tau^2 \dot{\kappa}) \mathbf{b}
 \end{aligned}$$

$$\begin{aligned}
 \text{Thus } [\dot{\mathbf{b}} \ddot{\mathbf{b}} \ddot{\mathbf{b}}] &= -\ddot{\mathbf{b}} \cdot (\dot{\mathbf{b}} \times \ddot{\mathbf{b}}) = (\dot{\tau} \mathbf{n} + \tau^2 \mathbf{b} - \kappa \tau \mathbf{t}) \cdot [3\tau^2 \dot{\mathbf{t}} + (2\tau \dot{\tau} \kappa + \tau^2 \dot{\kappa}) \mathbf{b}] \\
 &= \tau^3[-3\kappa \dot{\tau} + 2\kappa \dot{\tau} + \tau \dot{\kappa}] = \tau^3(\tau \dot{\kappa} - \kappa \dot{\tau})
 \end{aligned}$$

$$\text{and finally } \tau_3 = \frac{[\dot{\mathbf{b}} \ddot{\mathbf{b}} \ddot{\mathbf{b}}]}{|\dot{\mathbf{b}} \times \ddot{\mathbf{b}}|^2} = \frac{\tau^3(\tau \dot{\kappa} - \kappa \dot{\tau})}{\tau^4(\kappa^2 + \tau^2)} = \frac{\tau \dot{\kappa} - \kappa \dot{\tau}}{\tau(\kappa^2 + \tau^2)}$$

ENB

Supplementary Problems

- 4.26. Find the intersection of the x_1x_2 plane and the normal plane to the curve $\mathbf{x} = (\cos t)\mathbf{e}_1 + (\sin t)\mathbf{e}_2 + t\mathbf{e}_3$ at the point $t = \pi/2$. *Ans.* $x_1 = -\pi/2$, $x_2 = k$, $x_3 = 0$, $-\infty < k < \infty$
- 4.27. Find the intersection of the x_1x_2 plane and the tangent line to the curve $\mathbf{x} = (1+t)\mathbf{e}_1 - t^2\mathbf{e}_2 + (1+t^3)\mathbf{e}_3$ at $t = 1$. *Ans.* $(4/3, 1/3, 0)$
- 4.28. Show that a curve is a straight line if all tangent lines are parallel.
- 4.29. Let $\Delta\theta$ be the angle between the unit tangents $\mathbf{t}(s+\Delta s)$ and $\mathbf{t}(s)$, $\Delta s > 0$. Show that $\lim_{\Delta s \rightarrow 0} \Delta\theta = 0$.
- 4.30. Show that along the plane curve $\mathbf{x} = x_1(t)\mathbf{e}_1 + x_2(t)\mathbf{e}_2$ the curvature is
- $$|\kappa| = \frac{|x_1'x_2'' - x_2'x_1''|}{[(x_1')^2 + (x_2')^2]^{3/2}}$$
- 4.31. Show that a curve is a straight line if $\mathbf{x}'$ and $\mathbf{x}''$ are linearly dependent for all t .
- 4.32. Find the curvature along the curve $\mathbf{x} = (t - \sin t)\mathbf{e}_1 + (1 - \cos t)\mathbf{e}_2 + t\mathbf{e}_3$.
Ans. $|\kappa| = (1 + 4 \sin^4 t/2)^{1/2} / (1 + 4 \sin^2 t/2)^{3/2}$
- 4.33. Find the torsion along the curve $\mathbf{x} = (t - \sin t)\mathbf{e}_1 + (1 - \cos t)\mathbf{e}_2 + t\mathbf{e}_3$.
Ans. $\tau = -1/(1 + 4 \sin^4 t/2)$
- 4.34. Show that $\kappa\tau = |\dot{\mathbf{t}} \cdot \dot{\mathbf{b}}|$.
- 4.35. Show that the curve $\mathbf{x} = t\mathbf{e}_1 + \frac{1+t}{t}\mathbf{e}_2 + \frac{1-t^2}{t}\mathbf{e}_3$ lies in a plane.
- 4.36. Find the most general function $f(t)$ so that the curve $\mathbf{x} = a(\cos t)\mathbf{e}_1 + a(\sin t)\mathbf{e}_2 + f(t)\mathbf{e}_3$ will be a plane curve. *Ans.* $f(t) = A \sin t + B \cos t + C$, $A, B, C = \text{constant}$
- 4.37. Show that the osculating planes at any three points on the curve $\mathbf{x} = t\mathbf{e}_1 + \frac{1}{2}t^2\mathbf{e}_2 + \frac{1}{6}t^3\mathbf{e}_3$ meet at a point lying in the plane determined by these three points.
- 4.38. Show that $\mathbf{x} = \mathbf{x}(t)$ is a plane curve if and only if $[\mathbf{x}'\mathbf{x}''\mathbf{x}'''] = 0$.
- 4.39. Prove the converse of Theorem 4.4, page 70: If $\mathbf{x} = \mathbf{x}(s)$ is a plane curve, then $\tau \equiv 0$.
- 4.40. Show that a curve is a general helix iff the tangent indicatrix is a circle.
- 4.41. Show that the tangent to the tangent indicatrix of a curve C is parallel to the tangent to the binormal indicatrix at corresponding points.
- 4.42. Show that the curvature along the binormal indicatrix is $\kappa_3 = (\kappa^2 + \tau^2)/\kappa^2$.
- 4.43. Show that the torsion along the tangent indicatrix is $\tau_1 = \frac{\tau \dot{\kappa} - \kappa \dot{\tau}}{\kappa(\kappa^2 + \tau^2)}$.

Chapter 9

First and Second Fundamental Forms

FIRST FUNDAMENTAL FORM

We recall that a curve in E^3 is uniquely determined by two local invariant quantities, curvature and torsion, as functions of arc length. Similarly, a surface in E^3 is uniquely determined by certain local invariant quantities called the first and second fundamental forms.

Let $\mathbf{x} = \mathbf{x}(u, v)$ be a coordinate patch on a surface of class ∞^1 . We recall that the differential of the mapping $\mathbf{x} = \mathbf{x}(u, v)$ at (u, v) is a 1-1 linear mapping $d\mathbf{x} = \mathbf{x}_u du + \mathbf{x}_v dv$ of the vectors (du, dv) in the uv plane onto the vectors $\mathbf{x}_u du + \mathbf{x}_v dv$ parallel to the tangent plane at $\mathbf{x}(u, v)$, as shown in Fig. 9-1. Note that we are using the symbols du, dv both for the differentials of the coordinate functions in the uv plane and for the components of a typical vector in the uv plane. In the same spirit, we denote the image $\mathbf{x}_u du + \mathbf{x}_v dv$ simply by $d\mathbf{x}$. We recall further that $d\mathbf{x}$ has the property that

$$\mathbf{x}(u + du, v + dv) = \mathbf{x}(u, v) + d\mathbf{x} + o((du^2 + dv^2)^{1/2})$$

Thus the vector $d\mathbf{x}$ is a first order approximation to the vector $\mathbf{x}(u + du, v + dv) - \mathbf{x}(u, v)$ from the point $\mathbf{x}(u, v)$ on the patch to the neighboring point $\mathbf{x}(u + du, v + dv)$.

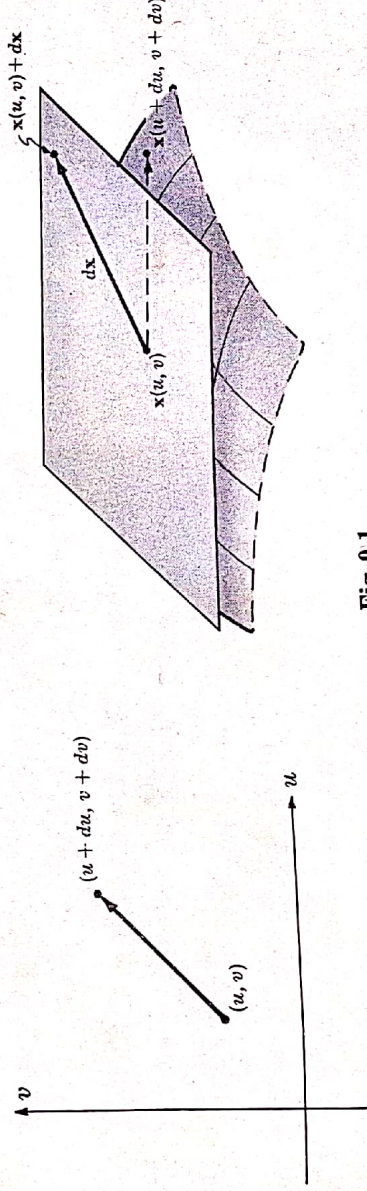


Fig. 9-1

We now consider the quantity

$$\begin{aligned} \mathbf{I} &= d\mathbf{x} \cdot d\mathbf{x} = (\mathbf{x}_u du + \mathbf{x}_v dv) \cdot (\mathbf{x}_u du + \mathbf{x}_v dv) \\ &= (\mathbf{x}_u \cdot \mathbf{x}_u) du^2 + 2(\mathbf{x}_u \cdot \mathbf{x}_v) du dv + (\mathbf{x}_v \cdot \mathbf{x}_v) dv^2 = E du^2 + 2F du dv + G dv^2 \end{aligned} \quad (9.1)$$

where we set

$$E = \mathbf{x}_u \cdot \mathbf{x}_u, \quad F = \mathbf{x}_u \cdot \mathbf{x}_v, \quad G = \mathbf{x}_v \cdot \mathbf{x}_v$$

The function $\mathbf{I} = d\mathbf{x} \cdot d\mathbf{x} = E du^2 + 2F du dv + G dv^2$ is called the *first fundamental form* of $\mathbf{x} = \mathbf{x}(u, v)$. It is a homogeneous function of second degree in du and dv with coefficients E, F and G , called the *first fundamental coefficients*, which are functions of u and v and vary from point to point on the coordinate patch. Thus the first fundamental form $\mathbf{I}$ is the quadratic form defined on vectors (du, dv) in the uv plane by

$$\mathbf{I}(du, dv) = E du^2 + 2F du dv + G dv^2$$

Recall that dx at (u, v) is the principal part of the vector from the point $x(u, v)$ on the patch to the point $x(u + du, v + dv)$. This suggests that in some sense I depends only on the surface and not on the particular representation. In fact this is true in the following sense. Suppose that $x = x^*(\theta, \phi)$ is another coordinate patch containing a neighborhood of $x(u, v)$. Then the parameter transformation $\theta = \theta(u, v)$, $\phi = \phi(u, v)$ has a differential at (u, v) which maps the vector (du, dv) into the vector $(d\theta, d\phi)$ given by $d\theta = \theta_u du + \theta_v dv$, $d\phi = \phi_u du + \phi_v dv$. The statement that I is independent of the representation is interpreted to mean that I and I^* agree under this correspondence; that is, $I(du, dv) = I^*(d\theta, d\phi)$. This can be verified analytically using the chain rule:

$$\begin{aligned} I^*(d\theta, d\phi) &= |dx^*|^2 = |(\mathbf{x}_\theta^* d\theta + \mathbf{x}_\phi^* d\phi)|^2 \\ &= |\mathbf{x}_\theta^*(\theta_u du + \theta_v dv) + \mathbf{x}_\phi^*(\phi_u du + \phi_v dv)|^2 \\ &= |(\mathbf{x}_\theta^* \theta_u + \mathbf{x}_\phi^* \phi_u) du + (\mathbf{x}_\theta^* \theta_v + \mathbf{x}_\phi^* \phi_v) dv|^2 \\ &= |\mathbf{x}_u du + \mathbf{x}_v dv|^2 = |dx|^2 = I(du, dv) \end{aligned}$$

The first fundamental coefficients themselves are not invariant under a parameter transformation, but instead they transform as follows:

$$\begin{aligned} E &= \mathbf{x}_u \cdot \mathbf{x}_u = (\mathbf{x}_\theta^* \theta_u + \mathbf{x}_\phi^* \phi_u) \cdot (\mathbf{x}_\theta^* \theta_u + \mathbf{x}_\phi^* \phi_u) \\ &= \mathbf{x}_\theta^* \cdot \mathbf{x}_\theta^* \theta_u^2 + 2\mathbf{x}_\theta^* \cdot \mathbf{x}_\phi^* \theta_u \phi_u + \mathbf{x}_\phi^* \cdot \mathbf{x}_\phi^* \phi_u^2 \\ &= E^* \theta_u^2 + 2F^* \theta_u \phi_u + G^* \phi_u^2 \end{aligned} \quad (9.2)$$

and similarly,

$$\begin{aligned} F &= E^* \theta_u \theta_v + F^* (\theta_u \phi_v + \phi_u \theta_v) + G^* \phi_u \phi_v \\ G &= E^* \theta_v^2 + 2F^* \theta_v \phi_v + G^* \phi_v^2 \end{aligned} \quad (9.3)$$

Finally, we note that the first fundamental form is *positive definite*. That is, $I \geq 0$, and $I = 0$ if and only if $du = 0$ and $dv = 0$. For, clearly $I = |dx|^2 \geq 0$, and, since $\mathbf{x}_u$ and $\mathbf{x}_v$ are independent, $I = |dx|^2 = 0$ if and only if $du = 0$ and $dv = 0$.

Since I is positive definite its coefficients must satisfy $E > 0$, $G > 0$ and $EG - F^2 > 0$. This can also be verified directly. For again, since $\mathbf{x}_u$ and $\mathbf{x}_v$ are independent, $\mathbf{x}_u \neq 0$, $\mathbf{x}_v \neq 0$, and hence $E = \mathbf{x}_u \cdot \mathbf{x}_u = |\mathbf{x}_u|^2 > 0$ and $G = \mathbf{x}_v \cdot \mathbf{x}_v = |\mathbf{x}_v|^2 > 0$. Also, using the vector identity, $[F_3]$, we find that

$$EG - F^2 = (\mathbf{x}_u \cdot \mathbf{x}_u)(\mathbf{x}_v \cdot \mathbf{x}_v) - (\mathbf{x}_u \cdot \mathbf{x}_v)(\mathbf{x}_v \cdot \mathbf{x}_u) = (\mathbf{x}_u \times \mathbf{x}_v) \cdot (\mathbf{x}_u \times \mathbf{x}_v) = |\mathbf{x}_u \times \mathbf{x}_v|^2 \quad (9.4)$$

But at every point $\mathbf{x}_u \times \mathbf{x}_v \neq 0$, and so $EG - F^2 > 0$.

Example 9.1.

Consider the surface given by

$$\mathbf{x} = (u + v)\mathbf{e}_1 + (u - v)\mathbf{e}_2 + uv\mathbf{e}_3$$

Here $\mathbf{x}_u = \mathbf{e}_1 + \mathbf{e}_2 + v\mathbf{e}_3$, $\mathbf{x}_v = \mathbf{e}_1 - \mathbf{e}_2 + u\mathbf{e}_3$, $E = \mathbf{x}_u \cdot \mathbf{x}_u = 2 + v^2$, $F = \mathbf{x}_u \cdot \mathbf{x}_v = uv$, $G = \mathbf{x}_v \cdot \mathbf{x}_v = 2 + u^2$ and

$$I = E du^2 + 2F du dv + G dv^2 = (2 + v^2) du^2 + 2uv du dv + (2 + u^2) dv^2$$

Observe that at all (u, v) we have $E > 0$, $G > 0$, and $EG - F^2 = 4 + 2u^2 + 2v^2 > 0$. If we introduce $\theta = u + v$, $\phi = u - v$, then the surface is also represented by

$$\mathbf{x} = \theta\mathbf{e}_1 + \phi\mathbf{e}_2 + \frac{1}{4}(\theta^2 - \phi^2)\mathbf{e}_3$$

Here $\mathbf{x}_\theta = \mathbf{e}_1 + \frac{1}{2}\theta\mathbf{e}_3$, $\mathbf{x}_\phi = \mathbf{e}_2 - \frac{1}{2}\phi\mathbf{e}_3$, $E^* = \mathbf{x}_\theta \cdot \mathbf{x}_\theta = 1 + \frac{1}{4}\theta^2$, $F^* = \mathbf{x}_\theta \cdot \mathbf{x}_\phi = -\frac{1}{4}\theta\phi$, $G^* = \mathbf{x}_\phi \cdot \mathbf{x}_\phi = 1 + \frac{1}{4}\phi^2$.

Observe that at the point corresponding to $u = 1$, $v = 1$ we have $E = 3$, $F = 1$, $G = 3$. But at the same point $\theta = 2$, $\phi = 0$, and $E^* = 2$, $F^* = 0$, $G^* = 1$. The first fundamental coefficients are not invariant quantities.

ARC LENGTH AND SURFACE AREA

The first fundamental coefficients play a basic role in calculating arc length, angles, and surface area. For, let $\mathbf{x} = \mathbf{x}(u(t), v(t))$, $a \leq t \leq b$, be a regular arc on a patch $\mathbf{x} = \mathbf{x}(u, v)$. We recall that its length is the integral

$$s = \int_a^b \left| \frac{d\mathbf{x}}{dt} \right| dt = \int_a^b \left(\frac{d\mathbf{x}}{dt} \cdot \frac{d\mathbf{x}}{dt} \right)^{1/2} dt = \int_a^b \left[\left(\mathbf{x}_u \frac{du}{dt} + \mathbf{x}_v \frac{dv}{dt} \right) \cdot \left(\mathbf{x}_u \frac{du}{dt} + \mathbf{x}_v \frac{dv}{dt} \right) \right]^{1/2} dt$$

Hence expanding,
$$s = \int_a^b \left[E \left(\frac{du}{dt} \right)^2 + 2F \frac{du}{dt} \frac{dv}{dt} + G \left(\frac{dv}{dt} \right)^2 \right]^{1/2} dt \quad (9.6)$$

Thus arc length on the surface is the integral of the square root of the first fundamental form.

Suppose further that $d\mathbf{x} = \mathbf{x}_u du + \mathbf{x}_v dv$ and $\delta \mathbf{x} = \mathbf{x}_u \delta u + \mathbf{x}_v \delta v$ are two vectors parallel to the tangent plane at a point $\mathbf{x}$. If α is the angle between $d\mathbf{x}$ and $\delta \mathbf{x}$, then

$$\begin{aligned} \cos \alpha &= \frac{d\mathbf{x} \cdot \delta \mathbf{x}}{|d\mathbf{x}| |\delta \mathbf{x}|} = \frac{(\mathbf{x}_u du + \mathbf{x}_v dv) \cdot (\mathbf{x}_u \delta u + \mathbf{x}_v \delta v)}{|\mathbf{x}_u du + \mathbf{x}_v dv| |\mathbf{x}_u \delta u + \mathbf{x}_v \delta v|} \\ &= \frac{E du \delta u + F(du \delta v + dv \delta u) + G dv \delta v}{[E du^2 + 2F du dv + G dv^2]^{1/2} [E \delta u^2 + 2F \delta u \delta v + G \delta v^2]^{1/2}} \end{aligned} \quad (9.6)$$

In particular, if β is the angle between the u - and v -parameter curves at $\mathbf{x}$, i.e. the angle between $\mathbf{x}_u$ and $\mathbf{x}_v$, then

$$\cos \beta = \frac{\mathbf{x}_u \cdot \mathbf{x}_v}{|\mathbf{x}_u| |\mathbf{x}_v|} = \frac{\mathbf{x}_u \cdot \mathbf{x}_v}{\sqrt{\mathbf{x}_u \cdot \mathbf{x}_u} \sqrt{\mathbf{x}_v \cdot \mathbf{x}_v}} = \frac{F}{\sqrt{EG}} \quad (9.7)$$

As a result of the above we have

Theorem 9.1.

(a) The tangent vectors $d\mathbf{x} = \mathbf{x}_u du + \mathbf{x}_v dv$ and $\delta \mathbf{x} = \mathbf{x}_u \delta u + \mathbf{x}_v \delta v$ are perpendicular if and only if

$$E du \delta u + F(du \delta v + dv \delta u) + G dv \delta v = 0$$

(b) The u - and v -parameter curves at a point are perpendicular if and only if $F = 0$.

Example 9.2.

Consider the image on the unit sphere

$$\mathbf{x} = (\cos \theta \sin \phi) \mathbf{e}_1 + (\sin \theta \sin \phi) \mathbf{e}_2 + (\cos \phi) \mathbf{e}_3$$

$$0 \leq t \leq \pi/2.$$

of the curve $\theta = \log \cot(\pi/4 - t/2)$, $\phi = \pi/2 - t$, the curve starts at the equator and As shown in Fig. 9-2, the curve starts at the north pole. To determine its winds like a spiral about the north pole. length we compute

$$\mathbf{x}_\theta = (-\sin \theta \sin \phi) \mathbf{e}_1 + (\cos \theta \sin \phi) \mathbf{e}_2$$

$$\mathbf{x}_\phi = (-\sin \theta \cos \phi) \mathbf{e}_2 - (\sin \phi) \mathbf{e}_3$$

$$\mathbf{x}_\phi = (\cos \theta \cos \phi) \mathbf{e}_1 + (\sin \theta \cos \phi) \mathbf{e}_2 + (\sin \phi) \mathbf{e}_3$$

$$\mathbf{x}_\phi = (\cos \theta \cos \phi) \mathbf{e}_1 + (\sin \theta \cos \phi) \mathbf{e}_2 + (\sin \phi) \mathbf{e}_3$$

$$F = \mathbf{x}_\theta \cdot \mathbf{x}_\phi = 0, \quad G = \mathbf{x}_\phi \cdot \mathbf{x}_\phi = 1$$

$$E = \mathbf{x}_\theta \cdot \mathbf{x}_\theta = \sin^2 \phi, \quad F = \mathbf{x}_\theta \cdot \mathbf{x}_\phi = 0, \quad G = \mathbf{x}_\phi \cdot \mathbf{x}_\phi = 1$$

$$\frac{d\theta}{dt} = \frac{\operatorname{cosec}^2(\pi/4 - t/2)}{2 \cot(\pi/4 - t/2)} = \frac{1}{2 \sin(\pi/4 - t/2) \cos(\pi/4 - t/2)} = \frac{1}{\sin(\pi/2 - t)}$$

$$\frac{d\phi}{dt} = \frac{\operatorname{cosec}^2(\pi/4 - t/2)}{2 \cot(\pi/4 - t/2)} = \frac{1}{2 \sin(\pi/4 - t/2) \cos(\pi/4 - t/2)} = \frac{1}{\sin(\pi/2 - t)}$$

$$\frac{d\phi}{dt} = \frac{\operatorname{cosec}^2(\pi/4 - t/2)}{2 \cot(\pi/4 - t/2)} = \frac{1}{2 \sin(\pi/4 - t/2) \cos(\pi/4 - t/2)} = \frac{1}{\sin(\pi/2 - t)}$$

$$\frac{d\phi}{dt} = \frac{\operatorname{cosec}^2(\pi/4 - t/2)}{2 \cot(\pi/4 - t/2)} = \frac{1}{2 \sin(\pi/4 - t/2) \cos(\pi/4 - t/2)} = \frac{1}{\sin(\pi/2 - t)}$$

$$\frac{d\phi}{dt} = \frac{\operatorname{cosec}^2(\pi/4 - t/2)}{2 \cot(\pi/4 - t/2)} = \frac{1}{2 \sin(\pi/4 - t/2) \cos(\pi/4 - t/2)} = \frac{1}{\sin(\pi/2 - t)}$$

$$\frac{d\phi}{dt} = \frac{\operatorname{cosec}^2(\pi/4 - t/2)}{2 \cot(\pi/4 - t/2)} = \frac{1}{2 \sin(\pi/4 - t/2) \cos(\pi/4 - t/2)} = \frac{1}{\sin(\pi/2 - t)}$$

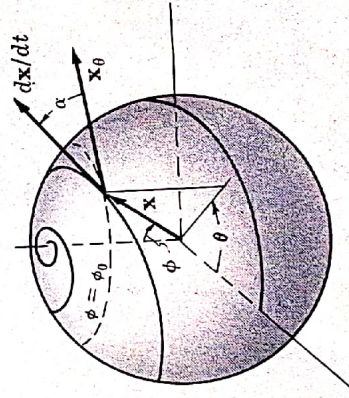


Fig. 9-2

$$\frac{d\phi}{dt} = -1 \quad \text{and} \quad \frac{1}{\sin(\pi/2 - t)}$$

$$\frac{d\theta}{dt} = \frac{\operatorname{cosec}^2(\pi/4 - t/2)}{2 \cot(\pi/4 - t/2)} = \frac{1}{2 \sin(\pi/4 - t/2) \cos(\pi/4 - t/2)} = \frac{1}{\sin(\pi/2 - t)}$$

$$\frac{d\theta}{dt} = \frac{\operatorname{cosec}^2(\pi/4 - t/2)}{2 \cot(\pi/4 - t/2)} = \frac{1}{2 \sin(\pi/4 - t/2) \cos(\pi/4 - t/2)} = \frac{1}{\sin(\pi/2 - t)}$$

$$\frac{d\theta}{dt} = \frac{\operatorname{cosec}^2(\pi/4 - t/2)}{2 \cot(\pi/4 - t/2)} = \frac{1}{2 \sin(\pi/4 - t/2) \cos(\pi/4 - t/2)} = \frac{1}{\sin(\pi/2 - t)}$$

$$\frac{d\theta}{dt} = \frac{\operatorname{cosec}^2(\pi/4 - t/2)}{2 \cot(\pi/4 - t/2)} = \frac{1}{2 \sin(\pi/4 - t/2) \cos(\pi/4 - t/2)} = \frac{1}{\sin(\pi/2 - t)}$$

$$\frac{d\theta}{dt} = \frac{\operatorname{cosec}^2(\pi/4 - t/2)}{2 \cot(\pi/4 - t/2)} = \frac{1}{2 \sin(\pi/4 - t/2) \cos(\pi/4 - t/2)} = \frac{1}{\sin(\pi/2 - t)}$$

$$\frac{d\theta}{dt} = \frac{\operatorname{cosec}^2(\pi/4 - t/2)}{2 \cot(\pi/4 - t/2)} = \frac{1}{2 \sin(\pi/4 - t/2) \cos(\pi/4 - t/2)} = \frac{1}{\sin(\pi/2 - t)}$$

and

Note that the curve makes a constant angle α with the parallels $\phi = \text{constant}$. For, using $\phi = \pi/2 - \psi$,

$$\begin{aligned}\cos \alpha &= \cos \angle \left(\frac{d\mathbf{x}}{dt}, \mathbf{x}_\theta \right) = \frac{\left(\mathbf{x}_\theta \frac{d\theta}{dt} + \mathbf{x}_\phi \frac{d\phi}{dt} \right) \cdot \mathbf{x}_\theta}{\left| \mathbf{x}_\theta \frac{d\theta}{dt} + \mathbf{x}_\phi \frac{d\phi}{dt} \right| |\mathbf{x}_\theta|} = \frac{E \frac{d\theta}{dt} + F \frac{d\phi}{dt}}{\sqrt{I} \sqrt{E}} \\ &= \left(\frac{\sin^2 \phi}{\sin(\pi/2 - t)} \right) \left(\frac{1}{\sqrt{2}} \frac{1}{\sin \phi} \right) = \frac{1}{\sqrt{2}} \quad 0 \leq t \leq \pi/2\end{aligned}$$

Now suppose ΔR is a small region on a patch bounded by neighboring v -parameter curves, u and $u + du$, and u -parameter curves, v and $v + dv$, as indicated in Fig. 9-3. As a first approximation to the area of ΔR we take the area of the parallelogram whose sides are the vectors $\Delta \mathbf{x}_1 = \mathbf{x}_u du$ and $\Delta \mathbf{x}_2 = \mathbf{x}_v dv$. Assuming $du > 0$ and $dv > 0$, this is the quantity

$$\Delta S = |\Delta \mathbf{x}_1 \times \Delta \mathbf{x}_2| = |\mathbf{x}_u \times \mathbf{x}_v| du dv = \sqrt{EG - F^2} du dv$$

Thus we are led to define the area of a region R on $\mathbf{x} = \mathbf{x}(u, v)$ as the double integral, if it exists,

$$A = \iint_W \sqrt{EG - F^2} du dv \quad (9.8)$$

where W is the set of points in the parameter plane which maps onto R .

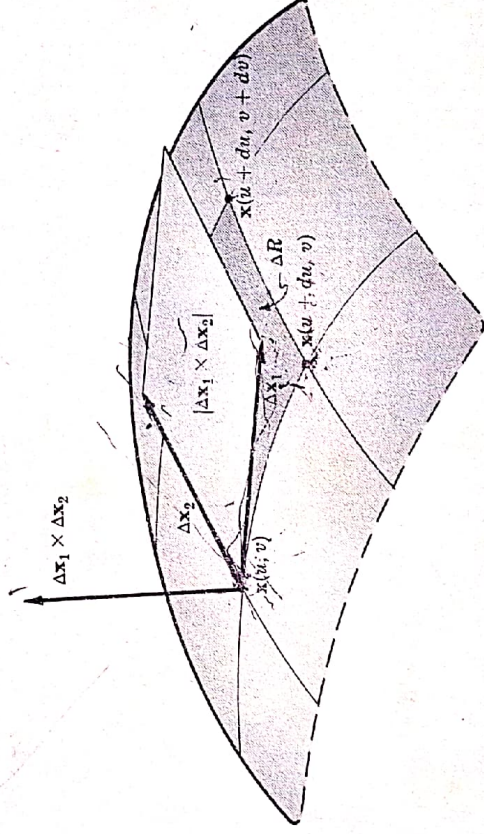


Fig. 9-3

Note that on an oriented surface the above definition of area is independent of the representation of R . For let $\mathbf{x} = \mathbf{x}^*(\theta, \phi)$ be any other patch containing R and such that $\partial(\theta, \phi)/\partial(u, v) > 0$ for all (u, v) in W . It can easily be computed from Equations (9.2) and (9.3) that

$$EG - F^2 = (E^*G^* - F^{*2})[\partial(\theta, \phi)/\partial(u, v)]^2$$

and so it follows from the transformation theorem for multiple integrals that

$$\begin{aligned}A &= \iint_W \sqrt{EG - F^2} du dv = \iint_W \sqrt{E^*G^* - F^{*2}} \frac{\partial(\theta, \phi)}{\partial(u, v)} du dv \\ &= \iint_{W^*} \sqrt{E^*G^* - F^{*2}} d\theta d\phi = A^*\end{aligned}$$

where W^* is the image of W in the $\theta\phi$ plane, which is the required result.

Example 9.3.

Consider the torus (see Example 8.5, page 158)

$$\mathbf{x} = (b + a \sin \phi)(\cos \theta)\mathbf{e}_1 + (b + a \sin \phi)(\sin \theta)\mathbf{e}_2 + (a \cos \phi)\mathbf{e}_3$$

Here $E = \mathbf{x}_\theta \cdot \mathbf{x}_\theta = (b + a \sin \phi)^2$, $F = \mathbf{x}_\theta \cdot \mathbf{x}_\phi = 0$, $G = \mathbf{x}_\phi \cdot \mathbf{x}_\phi = a^2$ and its surface area

$$S = \iint_{\substack{0 \leq \theta \leq 2\pi \\ 0 \leq \phi \leq 2\pi}} \sqrt{EG - F^2} d\theta d\phi = \int_0^{2\pi} \left[\int_0^{2\pi} a(b + a \sin \phi) d\phi \right] d\theta = 4\pi^2 ab$$

 $\sqrt{\text{SECOND FUNDAMENTAL FORM}}$

We now suppose $\mathbf{x} = \mathbf{x}(u, v)$ is a patch on a surface of class ≥ 2 . Then at each point on the patch there is a unit normal $\mathbf{N} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{|\mathbf{x}_u \times \mathbf{x}_v|}$, which is a function of u and v of class C^1 with differential $d\mathbf{N} = \mathbf{N}_u du + \mathbf{N}_v dv$. Observe in Fig. 9-4 that $d\mathbf{N}$ is orthogonal to $\mathbf{N}$ since it is parallel to the tangent plane of the spherical image of $\mathbf{N}$. This also follows from $0 = d(1) = d(\mathbf{N} \cdot \mathbf{N}) = 2d\mathbf{N} \cdot \mathbf{N}$. Thus $d\mathbf{N}$ is a vector parallel to the tangent plane at $\mathbf{x}$ as shown in Fig. 9-4(b).

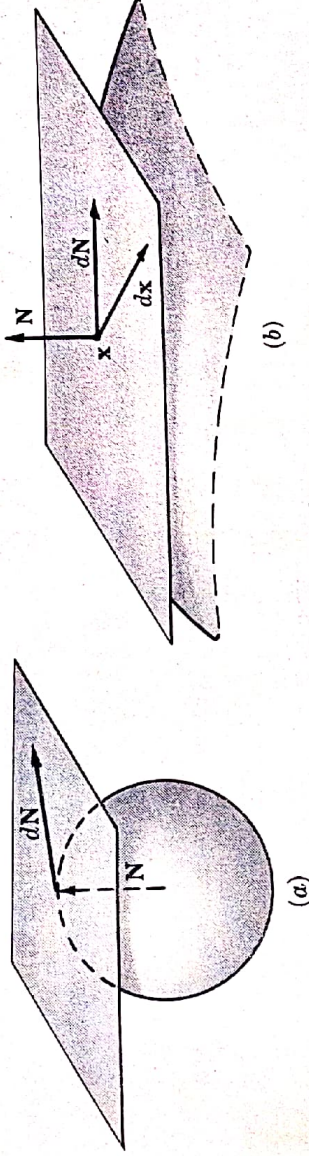


Fig. 9-4

Now consider the quantity

$$\begin{aligned} \text{II} &= -d\mathbf{x} \cdot d\mathbf{N} = -(\mathbf{x}_u du + \mathbf{x}_v dv) \cdot (\mathbf{N}_u du + \mathbf{N}_v dv) = L du^2 + 2M du dv + N dv^2. \\ &= -\mathbf{x}_u \cdot \mathbf{N}_u du^2 - (\mathbf{x}_u \cdot \mathbf{N}_v + \mathbf{x}_v \cdot \mathbf{N}_u) du dv - \mathbf{x}_v \cdot \mathbf{N}_v dv^2 = L du^2 + 2M du dv + N dv^2. \end{aligned} \quad (9.10)$$

where

$$L = -\mathbf{x}_u \cdot \mathbf{N}_u, \quad M = -\frac{1}{2}(\mathbf{x}_u \cdot \mathbf{N}_v + \mathbf{x}_v \cdot \mathbf{N}_u), \quad N = -\mathbf{x}_v \cdot \mathbf{N}_v$$

The function $\text{II} = -d\mathbf{x} \cdot d\mathbf{N} = L du^2 + 2M du dv + N dv^2$ is called the *second fundamental form* of $\mathbf{x} = \mathbf{x}(u, v)$. Here again II is a homogeneous function of second degree in du and dv with coefficients L , M and N , called the *second fundamental coefficients*, which are continuous functions of u and v .

It is easily shown that II is invariant in the same sense that I is invariant under a parameter transformation which preserves the direction of $\mathbf{N}$; otherwise II changes its sign. Also its coefficients transform in the same way as the first fundamental coefficients. That is, if $\mathbf{x} = \mathbf{x}^*(\theta, \phi)$ is any other patch on the surface, then at a point on the intersection where $\partial(\theta, \phi)/\partial(u, v) > 0$, we have

$$\begin{aligned} L &= L^* \theta_u^2 + 2M^* \theta_u \phi_u + N^* \phi_u^2 \\ M &= L^* \theta_u \theta_v + M^* (\phi_u \theta_v + \theta_u \phi_v) + N^* \phi_u \phi_v \\ N &= L^* \theta_v^2 + 2M^* \theta_v \phi_v + N^* \phi_v^2 \end{aligned} \quad (9.11)$$

where $\partial(\theta, \phi)/\partial(u, v) > 0$, we have

II is left to the reader as an exercise.

The proof of the above properties of II is left to the reader as an exercise.

Since $\mathbf{x}_u$ and $\mathbf{x}_v$ are perpendicular to $\mathbf{N}$ for all (u, v) , we have

$$\begin{aligned} 0 &= (\mathbf{x}_u \cdot \mathbf{N})_u = \mathbf{x}_{uu} \cdot \mathbf{N} + \mathbf{x}_u \cdot \mathbf{N}_u, & 0 &= (\mathbf{x}_u \cdot \mathbf{N})_v = \mathbf{x}_{uv} \cdot \mathbf{N} + \mathbf{x}_u \cdot \mathbf{N}_v, \\ 0 &= (\mathbf{x}_v \cdot \mathbf{N})_u = \mathbf{x}_{vu} \cdot \mathbf{N} + \mathbf{x}_v \cdot \mathbf{N}_u, & 0 &= (\mathbf{x}_v \cdot \mathbf{N})_v = \mathbf{x}_{vv} \cdot \mathbf{N} + \mathbf{x}_v \cdot \mathbf{N}_v. \end{aligned}$$

Hence

$$\mathbf{x}_{uu} \cdot \mathbf{N} = -\mathbf{x}_u \cdot \mathbf{N}_u, \quad \mathbf{x}_{uv} \cdot \mathbf{N} = -\mathbf{x}_u \cdot \mathbf{N}_v, \quad \text{and} \quad \mathbf{x}_{vv} \cdot \mathbf{N} = -\mathbf{x}_v \cdot \mathbf{N}_v.$$

This gives alternative expressions for L , M , and N . Namely from (9.10),

$$L = \mathbf{x}_{uu} \cdot \mathbf{N}, \quad M = \mathbf{x}_{uv} \cdot \mathbf{N}, \quad N = \mathbf{x}_{vv} \cdot \mathbf{N} \quad (9.12)$$

Also we have

$$\text{II} = L du^2 + 2M du dv + N dv^2 = \mathbf{x}_{uu} \cdot \mathbf{N} du^2 + 2\mathbf{x}_{uv} \cdot \mathbf{N} du dv + \mathbf{x}_{vv} \cdot \mathbf{N} dv^2 = d^2 \mathbf{x} \cdot \mathbf{N} \quad (9.13)$$

where $d^2 \mathbf{x} = \mathbf{x}_{uu} du^2 + 2\mathbf{x}_{uv} du dv + \mathbf{x}_{vv} dv^2$ is the second order derivative of $\mathbf{x}$ at (u, v) in the direction du, dv .

Example 9.4.

Consider the surface represented by

$$\mathbf{x} = ue_1 + ve_2 + (u^2 - v^2)e_3$$

$$\text{Here } \mathbf{x}_u = e_1 + 2ue_3, \quad \mathbf{x}_v = e_2 - 2ve_3, \quad \mathbf{x}_{uu} = 2e_3, \quad \mathbf{x}_{uv} = 0, \quad \mathbf{x}_{vv} = -2e_3$$

$$\mathbf{N} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{|\mathbf{x}_u \times \mathbf{x}_v|} = \frac{(4u^2 + 4v^2 + 1)^{-1/2}(-2ue_1 + 2ve_2 + e_3)}{\sqrt{(4u^2 + 4v^2 + 1)^{-1/2}(-2ue_1 + 2ve_2 + e_3)}}$$

Thus the second fundamental coefficients are

$$L = \mathbf{x}_{uu} \cdot \mathbf{N} = 2(4u^2 + 4v^2 + 1)^{-1/2}, \quad M = \mathbf{x}_{uv} \cdot \mathbf{N} = 0, \quad N = \mathbf{x}_{vv} \cdot \mathbf{N} = -2(4u^2 + 4v^2 + 1)^{-1/2}$$

and the second fundamental form is

$$\text{II} = L du^2 + 2M du dv + N dv^2 = 2(4u^2 + 4v^2 + 1)^{-1/2}(du^2 - dv^2)$$

Suppose P is a point on a surface of class ≥ 2 , Q is a point in the neighborhood of P and $\mathbf{x} = \mathbf{x}(u, v)$ is a patch containing P and Q . Let $d = \mathbf{PQ} \cdot \mathbf{N}$ be the projection of $\mathbf{PQ}$ onto the unit normal $\mathbf{N}$ at P , as shown in Fig. 9-5. Observe that d is positive or negative depending on whether Q is on one or the other side of the tangent plane at P and that $|d|$ is the perpendicular distance from Q to the tangent plane at P . Now suppose P and Q are the points $\mathbf{x}(u, v)$ and $\mathbf{x}(u + du, v + dv)$ respectively. Taylor's theorem gives

$$\mathbf{x}(u + du, v + dv) = \mathbf{x}(u, v) + d\mathbf{x} + \frac{1}{2}d^2\mathbf{x} + o(du^2 + dv^2)$$

$$\begin{aligned} \text{Thus } d &= \mathbf{PQ} \cdot \mathbf{N} = (\mathbf{x}(u + du, v + dv) - \mathbf{x}(u, v)) \cdot \mathbf{N} \\ &= [d\mathbf{x} + \frac{1}{2}d^2\mathbf{x} + o(du^2 + dv^2)] \cdot \mathbf{N} \\ &= d\mathbf{x} \cdot \mathbf{N} + \frac{1}{2}d^2\mathbf{x} \cdot \mathbf{N} + o(du^2 + dv^2) \end{aligned}$$

But $d\mathbf{x} \cdot \mathbf{N} = 0$, since $d\mathbf{x}$ is parallel to the tangent plane at P . Hence from (9.13),

$$d = \frac{1}{2}d^2\mathbf{x} \cdot \mathbf{N} + o(du^2 + dv^2) = \frac{1}{2}\text{II} + o(du^2 + dv^2)$$

Thus II is the principal part of twice the projection of $\mathbf{PQ}$ onto $\mathbf{N}$ and $|\text{II}|$ is the principal part of twice the perpendicular distance from Q onto the tangent plane at P .

The function

$$\delta = \frac{1}{2}\text{II} = \frac{1}{2}(L du^2 + 2M du dv + N dv^2)$$

is called the *osculating paraboloid* at P . The nature of this paraboloid determines qualitatively the nature of the surface in the neighborhood of P . We distinguish four cases depending upon the discriminant $LN - M^2$.

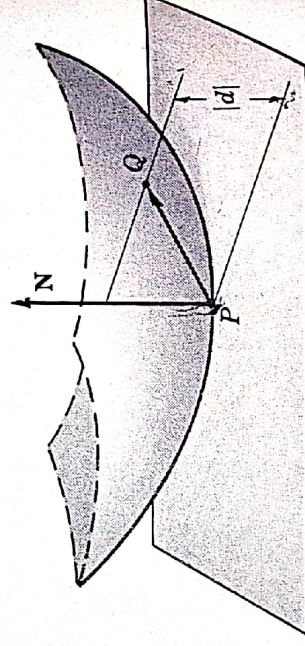


Fig. 9-5

- (i) *Elliptic case:* A point is called an *elliptic point* if $LN - M^2 > 0$. In this case δ as a function of du and dv is an elliptic paraboloid as shown in Fig. 9-6(a). Observe that δ maintains the same sign for all (du, dv) . In the neighborhood of an elliptic point the surface lies on one side of the tangent plane at the point and is shaped like the figure.

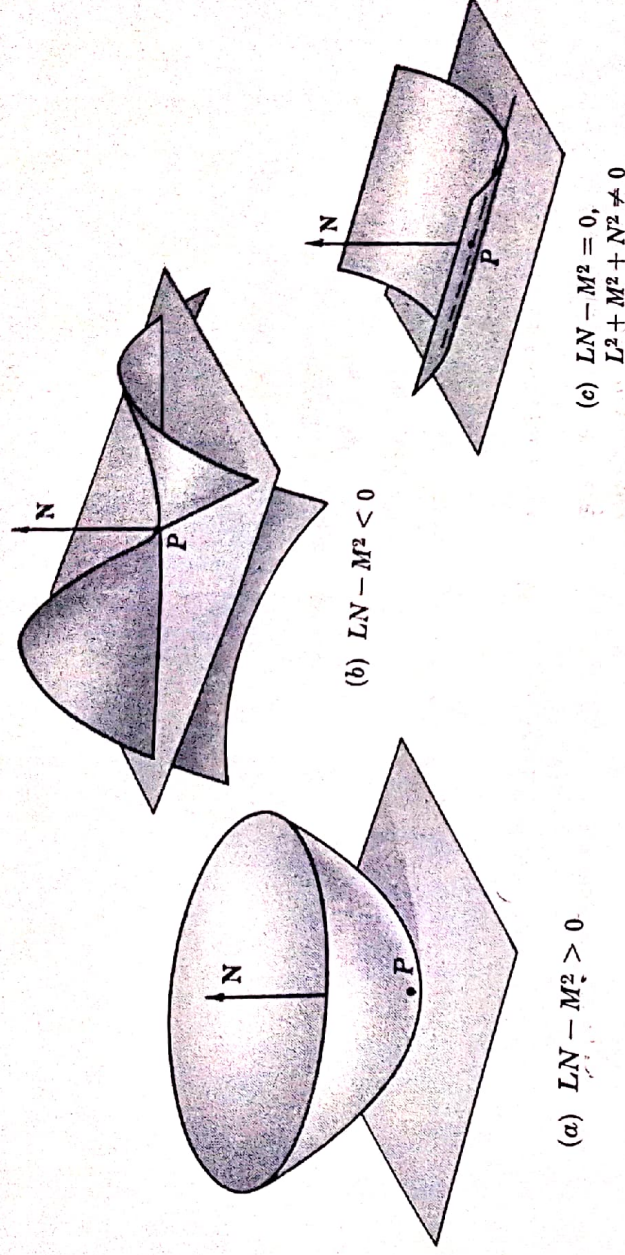


Fig. 9-6

- (ii) *Hyperbolic case:* A point is called a *hyperbolic point* if $LN - M^2 < 0$. In this case δ as a function of (du, dv) is a hyperbolic paraboloid as shown in Fig. 9-6(b). Here there are two distinct lines in the tangent plane through P which divide the tangent plane into four sections in which δ is alternately positive and negative. On the two lines, $\delta = 0$. In the neighborhood of a hyperbolic point the surface lies on both sides of the tangent plane as in the figure.
- (iii) *Parabolic case:* A point is called a *parabolic point* if $LN - M^2 = 0$ and $L^2 + M^2 + N^2 \neq 0$, i.e. if $LN - M^2 = 0$ and the coefficients L , M and N are not all zero. In this case δ as a function of (du, dv) is a parabolic cylinder, as shown in Fig. 9-6(c). Here there is a single line in the tangent plane through P along which $\delta = 0$, otherwise δ maintains the same sign. It is to be noted that in the neighborhood of a parabolic point the surface itself may lie on both sides of the tangent plane. See Problem 9.8.
- (iv) *Planar case:* A point is called a *planar point* if $L = M = N = 0$. Here $\delta = 0$ for all (du, dv) . In this case the degree of *contact* of the surface and the tangent plane is of higher order than in the preceding cases.
- We expect from the above geometric discussion that the property of a point on a surface being elliptic, hyperbolic, parabolic or planar is independent of the representation of the surface. This can be verified analytically. For if $\mathbf{x} = \mathbf{x}^*(\theta, \phi)$ is any other patch on the surface. This can be verified from equations (9.11) that at each point on the intersection surface, it can be shown from equations (9.11) that
- $$L^*N^* - M^{*2} = \left(\frac{\partial(u, v)}{\partial(\theta, \phi)} \right)^2 (LN - M^2)$$
- Since $\partial(u, v)/\partial(\theta, \phi) \neq 0$, it follows that $L^*N^* - M^{*2}$ is positive, negative, or zero together with $LN - M^2$. It also follows from equations (9.11) and their corresponding inverse equations that $L = M = N = 0$ if and only if $L^* = M^* = N^* = 0$.

Example 9.5.

We refer to the torus

$$\mathbf{x} = (b + a \sin \phi)(\cos \theta)\mathbf{e}_1 + (b + a \sin \phi)(\sin \theta)\mathbf{e}_2 + (a \cos \phi)\mathbf{e}_3 \quad b > a$$

Here

$$\mathbf{x}_{\theta\theta} = -(b + a \sin \phi)(\cos \theta)\mathbf{e}_1 - (b + a \sin \phi)(\sin \theta)\mathbf{e}_2 + (a \cos \phi)\mathbf{e}_3 \quad \checkmark$$

$$\mathbf{x}_{\theta\phi} = -(a \cos \phi \sin \theta)\mathbf{e}_1 + (a \cos \phi \cos \theta)\mathbf{e}_2 \quad \checkmark$$

$$\mathbf{x}_{\phi\phi} = -(a \sin \phi \cos \theta)\mathbf{e}_1 - (a \sin \phi \sin \theta)\mathbf{e}_2 - (a \cos \phi)\mathbf{e}_3$$

$$\mathbf{N} = (-\cos \theta \sin \phi)\mathbf{e}_1 - (\sin \theta \sin \phi)\mathbf{e}_2 - (\cos \phi)\mathbf{e}_3$$

$$L = \mathbf{x}_{\theta\theta} \cdot \mathbf{N} = (b + a \sin \phi) \sin \phi \quad M = \mathbf{x}_{\theta\phi} \cdot \mathbf{N} = 0 \quad N = \mathbf{x}_{\phi\phi} \cdot \mathbf{N} = a$$

and

$$LN - M^2 = a(b + a \sin \phi) \sin \phi$$

Observe that the second fundamental coefficients depend only on ϕ . They are constant along a parallel $\phi = \phi_0$. Since $0 < a < b$, $a(b + a \sin \phi) > 0$. Hence the sign of $LN - M^2$ agrees with the sign of $\sin \phi$. Thus $LN - M^2 > 0$ for $0 < \phi < \pi$, $LN - M^2 = 0$ for $\phi = 0$ or $\phi = \pi$, and $LN - M^2 < 0$ for $\pi < \phi < 2\pi$. Thus as shown in Fig. 9-7, the points towards the outside of the torus ($0 < \phi < \pi$) are elliptic. Here the surface lies on one side of the tangent plane in the neighborhood of each point. The points on the inside of the torus ($\pi < \phi < 2\pi$) are hyperbolic. Here the surface lies on both sides of the tangent plane in the neighborhood of each point. The top and bottom parallels of torus, $\phi = 0$ and $\phi = \pi$ are parabolic points.

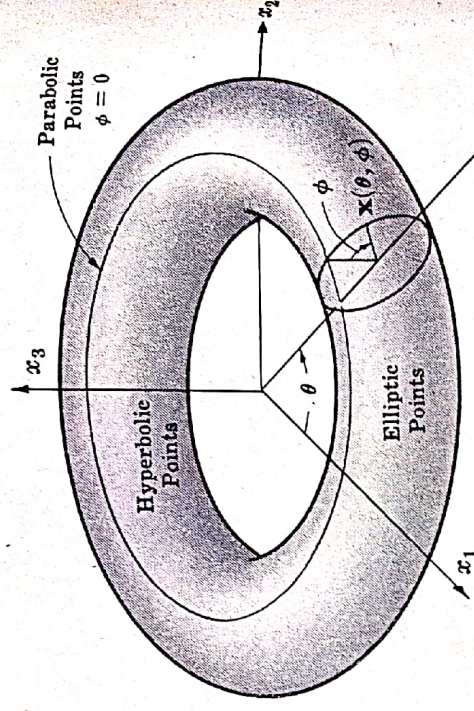


Fig. 9-7

The nature of the surface at a planar point is described by the next higher order terms of the expansion of $\mathbf{x}(u, v)$. Namely, suppose $\mathbf{x} = \mathbf{x}(u, v)$ is of class C^3 and $P = \mathbf{x}(u, v)$ is a planar point; then

$$\mathbf{x}(u + du, v + dv) = \mathbf{x}(u, v) + d\mathbf{x} + \frac{1}{2}d^2\mathbf{x} + \frac{1}{6}d^3\mathbf{x} + o[(du^2 + dv^2)^{3/2}]$$

and

$$\begin{aligned} d &= [\mathbf{x}(u + du, v + dv) - \mathbf{x}(u, v)] \cdot \mathbf{N} \\ &= d\mathbf{x} \cdot \mathbf{N} + \frac{1}{2}d^2\mathbf{x} \cdot \mathbf{N} + \frac{1}{6}d^3\mathbf{x} \cdot \mathbf{N} + o[(du^2 + dv^2)^{3/2}] \\ &= \frac{1}{2}II + \frac{1}{6}d^3\mathbf{x} \cdot \mathbf{N} + o[(du^2 + dv^2)^{3/2}] \end{aligned}$$

Since $L = M = N = 0$ at P , we have $II = 0$. Hence the surface is approximated by

$$\begin{aligned} \delta &= \frac{1}{6}d^3\mathbf{x} \cdot \mathbf{N} = \frac{1}{6}[\mathbf{x}_{uuu} \cdot \mathbf{N} du^3 + 3\mathbf{x}_{uuv} \cdot \mathbf{N} du^2 dv + 3\mathbf{x}_{uvv} \cdot \mathbf{N} du dv^2 + \mathbf{x}_{vvv} \cdot \mathbf{N} dv^3] \\ &= \frac{1}{6}[A du^3 + B du^2 dv + C du dv^2 + D dv^3] \end{aligned}$$

If the cubic $Ax^3 + Bx^2 + Cx + D$ has three distinct roots, then there are three lines in the tangent plane through P which divide the tangent plane at P into six sectors on which δ is alternately positive and negative. In this case δ as a function of (du, dv) describes a *monkey saddle* as shown in Fig. 9-8. Variations will occur depending on the nature of the roots of $Ax^3 + Bx^2 + Cx + D$.

Example 9.6.

Consider the surface

$$\mathbf{x} = u\mathbf{e}_1 + v\mathbf{e}_2 + (u^3 + v^3 + uv^4)\mathbf{e}_3$$

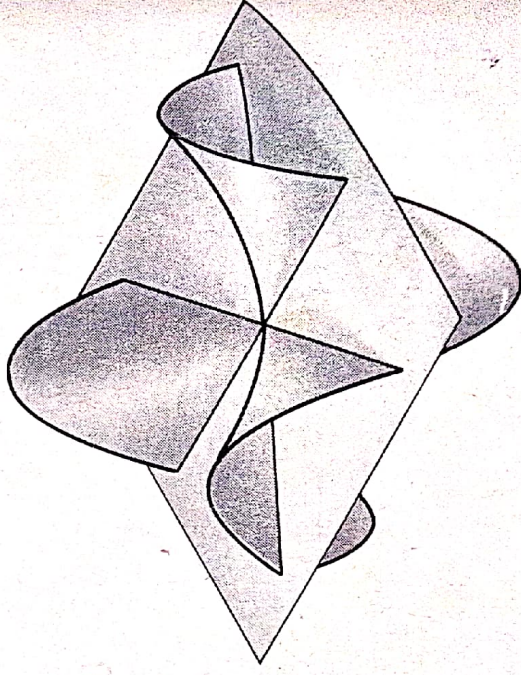


Fig. 9-8

$$\text{Here } \mathbf{x}_u = \mathbf{e}_1 + (3u^2 + 4u^3)\mathbf{e}_3, \quad \mathbf{x}_v = \mathbf{e}_2 + 3v^2\mathbf{e}_3, \\ \mathbf{N} = [(3u^2 + 4u^3)^2 + 9v^4 + 1]^{-1/2}(-3u^2 + 4u^3)\mathbf{e}_1 - 3v^2\mathbf{e}_2 + 6v\mathbf{e}_3 \\ \mathbf{x}_{uuu} = (6 + 24u)\mathbf{e}_3, \quad \mathbf{x}_{uuv} = \mathbf{0}, \quad \mathbf{x}_{uvv} = \mathbf{0}, \quad \mathbf{x}_{vvv} = 6v\mathbf{e}_3$$

At $u = 0, v = 0$, we have $\mathbf{x}_{uu} = \mathbf{x}_{uv} = \mathbf{x}_{vv} = \mathbf{0}$ so that $L = M = N = 0$ and the point is planar. The surface is described at this point by

$$\delta = \frac{1}{6}(\mathbf{x}_{uuu} \cdot \mathbf{N}) du^3 + 3\mathbf{x}_{uuv} \cdot \mathbf{N} du^2 dv + 3\mathbf{x}_{uvv} \cdot \mathbf{N} du dv^2 + \mathbf{x}_{vvv} \cdot \mathbf{N} dv^3$$

$$\delta = \frac{1}{6}[6\mathbf{e}_3 \cdot \mathbf{e}_3] du^3 + 6\mathbf{e}_3 \cdot \mathbf{e}_3 dv^3 \\ = du^3 + dv^3$$

$$= (du + dv)(du^2 - du dv + dv^2)$$

Here there is a single line, $du + dv = 0$, in the tangent plane along which $\delta = 0$. The second factor $(du^2 - du dv + dv^2)$ is positive definite for all real (du, dv) . In the neighborhood of this point the surface is shaped as shown in Fig. 9-9.

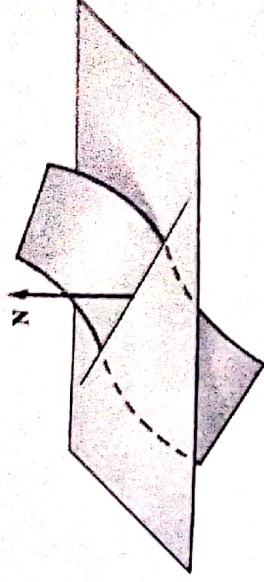


Fig. 9-9

NORMAL CURVATURE

Let P be a point on a surface of class ≥ 2 , $\mathbf{x} = \mathbf{x}(u, v)$ a patch containing P , and $\mathbf{x} = \mathbf{x}(u(t), v(t))$ a regular curve C of class C^2 through P . The *normal curvature vector* to C at P , denoted by $\mathbf{k}_n$, is the vector projection of the curvature vector $\mathbf{k}$ of C at P onto the normal $\mathbf{N}$ at P ; i.e.,

$$\mathbf{k}_n = (\mathbf{k} \cdot \mathbf{N})\mathbf{N} \quad (9.14)$$

Note that $\mathbf{k}_n$ is independent of the sense of $\mathbf{N}$. It is also independent of the sense of C since $\mathbf{k}$ is independent of the sense of C .

The component of $\mathbf{k}_n$ in the direction of $\mathbf{N}$ is called the *normal curvature* of C at P and is denoted by κ_n . That is,

$$\kappa_n = \mathbf{k} \cdot \mathbf{N} \quad (9.15)$$

Here the sign of κ_n depends upon the sense of $\mathbf{N}$; but it is independent of the sense of C .

We recall that the unit tangent to C at P is the vector $\mathbf{t} = \frac{d\mathbf{x}}{ds} = \frac{d\mathbf{x}}{dt} / \left| \frac{d\mathbf{x}}{dt} \right|$, and the curvature vector is $\mathbf{k} = \frac{d\mathbf{t}}{ds} = \frac{d}{dt} \left(\frac{d\mathbf{x}}{dt} / \left| \frac{d\mathbf{x}}{dt} \right| \right)$. Thus using the fact that $\mathbf{t}$ is perpendicular to

$\mathbf{N}$ along the curve, so that $0 = \frac{d}{dt}(\mathbf{t} \cdot \mathbf{N}) = \frac{d\mathbf{t}}{dt} \cdot \mathbf{N} + \mathbf{t} \cdot \frac{d\mathbf{N}}{dt}$, we have

$$\begin{aligned} \kappa_n &= \mathbf{k} \cdot \mathbf{N} = \frac{d\mathbf{t}}{dt} \cdot \mathbf{N} / \left| \frac{d\mathbf{x}}{dt} \right| = -\mathbf{t} \cdot \frac{d\mathbf{N}}{dt} / \left| \frac{d\mathbf{x}}{dt} \right| \\ &= -\frac{d\mathbf{x}}{dt} \cdot \frac{d\mathbf{N}}{dt} / \left| \frac{d\mathbf{x}}{dt} \right|^2 = -\frac{d\mathbf{x}}{dt} \cdot \frac{d\mathbf{N}}{dt} / \frac{d\mathbf{x}}{dt} \cdot \frac{d\mathbf{x}}{dt} \\ &= -\left(\frac{du}{dt} + \mathbf{x}_u \frac{dv}{dt} \right) \cdot \left(\frac{du}{dt} + \mathbf{x}_u \frac{dv}{dt} \right) / \left(\frac{du}{dt} + \mathbf{x}_u \frac{dv}{dt} \right) \cdot \left(\frac{du}{dt} + \mathbf{x}_u \frac{dv}{dt} \right) \end{aligned} \quad (9.16)$$

Hence

$$\kappa_n = \frac{L(du/dt)^2 + 2M(du/dt)(dv/dt) + N(dv/dt)^2}{E(du/dt)^2 + 2F(du/dt)(dv/dt) + G(dv/dt)^2}$$

Observe that κ_n as function of du/dt and dv/dt depends only upon the ratio $(du/dt)/(dv/dt)$, i.e. the *direction of the tangent line to C at P* . Otherwise κ_n is a function of the first and second fundamental coefficients which depend only upon P . Thus

Theorem 9.2. All curves through a point P on a patch which are tangent to the same line through P have the same normal curvature at P .

Suppose now C is a curve with a continuous principal normal $\mathbf{n}$ at P , and suppose the sense of $\mathbf{n}$ along C is chosen so that $0 \leq \angle(\mathbf{n}, \mathbf{N}) \leq \pi/2$ at P . From equation (9.15),

$$\kappa_n = \mathbf{k} \cdot \mathbf{N} = \mathbf{i} \cdot \mathbf{N} = \kappa(\mathbf{n} \cdot \mathbf{N}) = \kappa \cos \alpha \quad (9.17)$$

where $\alpha = \angle(\mathbf{n}, \mathbf{N})$. Since κ_n depends only upon the direction of the tangent to C and $\cos \alpha$ is determined by the direction of the principal normal line to C , it follows that if $\cos \alpha \neq 0$, the curvature κ of C at P is uniquely determined by the osculating plane to C , as shown in Fig. 9-10.

Note that $\cos \alpha = 0$ if and only if $\mathbf{n}$ is parallel to the tangent plane at P or, equivalently, if and only if the osculating and tangent planes coincide. Thus we have

Theorem 9.3. All curves on a surface through a point P which have the same osculating plane through P have the same curvature κ at P provided that the osculating plane is not tangent to the surface.

It follows from the above theorem that except for curves whose osculating plane is tangent to the surface, all possible values for the curvature of a curve through P can be obtained by considering the curves cut out by passing planes through P . In particular suppose C is a curve through P cut out by a plane containing $\mathbf{N}$, i.e. suppose C is a normal section of the patch. Then $\mathbf{n} \cdot \mathbf{N} = 1$ and from (9.17) we have $\kappa_n = \kappa$. Hence

Theorem 9.4. The curvature of a normal section of a patch at a point P is equal to the normal curvature of the section at P .

Since the normal curvature to C at P depends only upon P and the direction of the tangent line to C at P , we can speak of the normal curvature at P in the direction $d\mathbf{u}:dv$, $d\mathbf{u}^2 + dv^2 \neq 0$, and write, from (9.16),

$$\kappa_n = \frac{L d\mathbf{u}^2 + 2M d\mathbf{u} dv + N dv^2}{E d\mathbf{u}^2 + 2F d\mathbf{u} dv + G dv^2} = \frac{\text{II}}{\text{I}} \quad (9.18)$$

Here $d\mathbf{u}:dv$ are the *direction numbers* of the line in the tangent plane parallel to vector $\mathbf{x}_u d\mathbf{u} + \mathbf{x}_v dv$. Direction numbers $d\mathbf{u}:dv$ and $d\mathbf{u}':dv'$ determine the same line if and only if they are proportional, i.e. if and only if there exists $\lambda \neq 0$ such that $d\mathbf{u} = \lambda d\mathbf{u}'$ and $dv = \lambda dv'$.

From equation (9.18) it follows that κ_n is invariant (in the same sense as I and II) under a parametric transformation which preserves the sense of $\mathbf{N}$ and that κ_n changes sign under a parametric transformation which reverses the sense of $\mathbf{N}$. Also, since I is positive definite, it follows that κ_n is positive, negative, or zero together with II. If P is an elliptic point, $\kappa_n \neq 0$ and maintains the same sign for all $d\mathbf{u}:dv$. If P is hyperbolic, κ_n is positive, negative, or zero, depending on $d\mathbf{u}:dv$. If P is parabolic, κ_n maintains the same sign and is zero for the one direction for which $\text{II} = 0$. At a planar point, $\kappa_n = 0$ in all directions.

Example 9.7.

Consider the sphere of radius a ,

$$\mathbf{x} = (a \cos \theta \sin \phi) \mathbf{e}_1 + (a \sin \theta \sin \phi) \mathbf{e}_2 + (a \cos \phi) \mathbf{e}_3 \quad 0 < \theta < 2\pi, \quad 0 < \phi < \pi$$

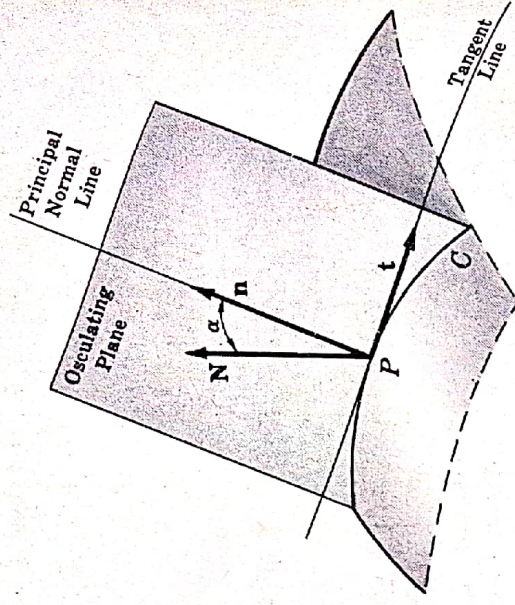


Fig. 9-10

Here

$$\begin{aligned} \mathbf{x}_\theta &= -(a \sin \theta \sin \phi) \mathbf{e}_1 + (a \cos \theta \sin \phi) \mathbf{e}_2 \\ \mathbf{x}_\phi &= (a \cos \theta \cos \phi) \mathbf{e}_1 + (a \sin \theta \cos \phi) \mathbf{e}_2 - (a \sin \phi) \mathbf{e}_3 \\ \mathbf{x}_{\theta\theta} &= -(a \cos \theta \sin \phi) \mathbf{e}_1 - (a \sin \theta \sin \phi) \mathbf{e}_2 \\ \mathbf{x}_{\theta\phi} &= -(a \sin \theta \cos \phi) \mathbf{e}_1 + (a \cos \theta \cos \phi) \mathbf{e}_2 \\ \mathbf{x}_{\phi\phi} &= -(a \cos \theta \sin \phi) \mathbf{e}_1 - (a \sin \theta \sin \phi) \mathbf{e}_2 - (a \cos \phi) \mathbf{e}_3 \\ N &= -(\cos \theta \sin \phi) \mathbf{e}_1 - (\sin \theta \sin \phi) \mathbf{e}_2 - (\cos \phi) \mathbf{e}_3 \\ E &= \mathbf{x}_\theta \cdot \mathbf{x}_\theta = a^2 \sin^2 \phi, \quad F = \mathbf{x}_\theta \cdot \mathbf{x}_\phi = 0, \quad G = \mathbf{x}_\phi \cdot \mathbf{x}_\phi = a^2 \\ L &= \mathbf{x}_{\theta\theta} \cdot \mathbf{N} = a \sin^2 \phi, \quad M = \mathbf{x}_{\theta\phi} \cdot \mathbf{N} = 0, \quad N = \mathbf{x}_{\phi\phi} \cdot \mathbf{N} = a \\ \kappa_n &= \frac{L d\theta^2 + 2M d\theta d\phi + N d\phi^2}{E d\theta^2 + 2F d\theta d\phi + G d\phi^2} = \frac{a \sin^2 \phi d\theta^2 + a d\phi^2}{a^2 \sin^2 \phi d\theta^2 + a^2 d\phi^2} = 1/a \end{aligned}$$

Here $\kappa_n = \text{constant} = 1/a$ at every point and in every direction. This is consistent with the well-known fact that every normal section of the sphere at every point intersects the sphere in a great circle with radius a and curvature $1/a$.

Example 9.8.

Consider the surface

$$\mathbf{x} = u\mathbf{e}_1 + v\mathbf{e}_2 + (u^2 - v^2)\mathbf{e}_3$$

Here

$$\begin{aligned} \mathbf{x}_u &= \mathbf{e}_1 + 2u\mathbf{e}_3, \quad \mathbf{x}_v = \mathbf{e}_2 - 2v\mathbf{e}_3, \quad \mathbf{x}_{uu} = 2\mathbf{e}_3, \quad \mathbf{x}_{uv} = 0, \quad \mathbf{x}_{vv} = -2\mathbf{e}_3 \\ N &= (4u^2 + 4v^2 + 1)^{-1/2}(-2u\mathbf{e}_1 + 2v\mathbf{e}_2 + \mathbf{e}_3), \quad E = 1 + 4u^2, \quad F = -4uv, \quad G = 1 + 4v^2 \end{aligned}$$

$$L = 2(4u^2 + 4v^2 + 1)^{-1/2}, \quad M = 0, \quad N = -2(4u^2 + 4v^2 + 1)^{-1/2}, \quad LN - M^2 = -4(4u^2 + 4v^2 + 1)^{-1}$$

Observe that every point is hyperbolic. In particular, at the origin we have $E = 1$, $F = 0$, $G = 1$, $L = 2$, $M = 0$, $N = -2$ so that $\kappa_n = \frac{2(du^2 - dv^2)}{du^2 + dv^2}$. If we assume $du^2 + dv^2 = 1$ and set $du = \cos \theta$, $dv = \sin \theta$ we have $\kappa_n = 2(\cos^2 \theta - \sin^2 \theta) = 2 \cos 2\theta$. Thus as shown in Fig. 9-11, κ_n varies between

positive and negative values in the four intervals $-\pi/4 \leq \theta \leq \pi/4$, $\pi/4 \leq \theta \leq 3\pi/4$, $3\pi/4 \leq \theta \leq 5\pi/4$, $5\pi/4 \leq \theta \leq 7\pi/4$. The curvature κ of a normal section through the origin will vary as κ_n itself, between -2 and 2 .

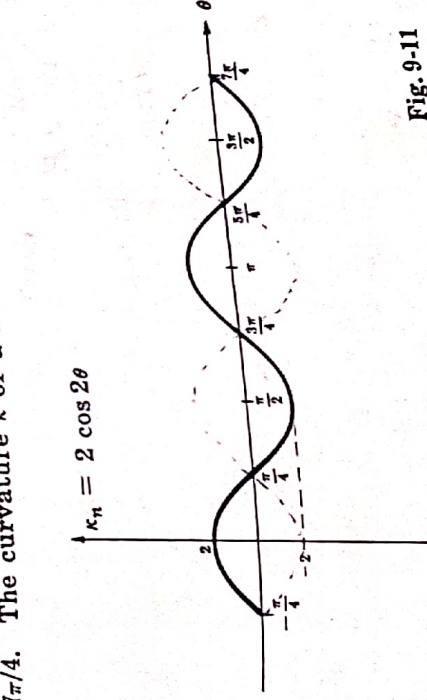


Fig. 9-11

PRINCIPAL CURVATURES AND DIRECTIONS

We want to investigate the normal curvature at point P on a surface in detail. Because of the invariant properties of κ_n , we can assume that a neighborhood of the form represented by a Monge patch that $\mathbf{x}_u = \mathbf{e}_1$ and $\mathbf{x}_v = \mathbf{e}_2 + f(u, v)\mathbf{e}_3$. This is accomplished by positioning the surface so that P is at the origin and so that the tangent plane coincides with the xy -plane as shown in Fig. 9-12.

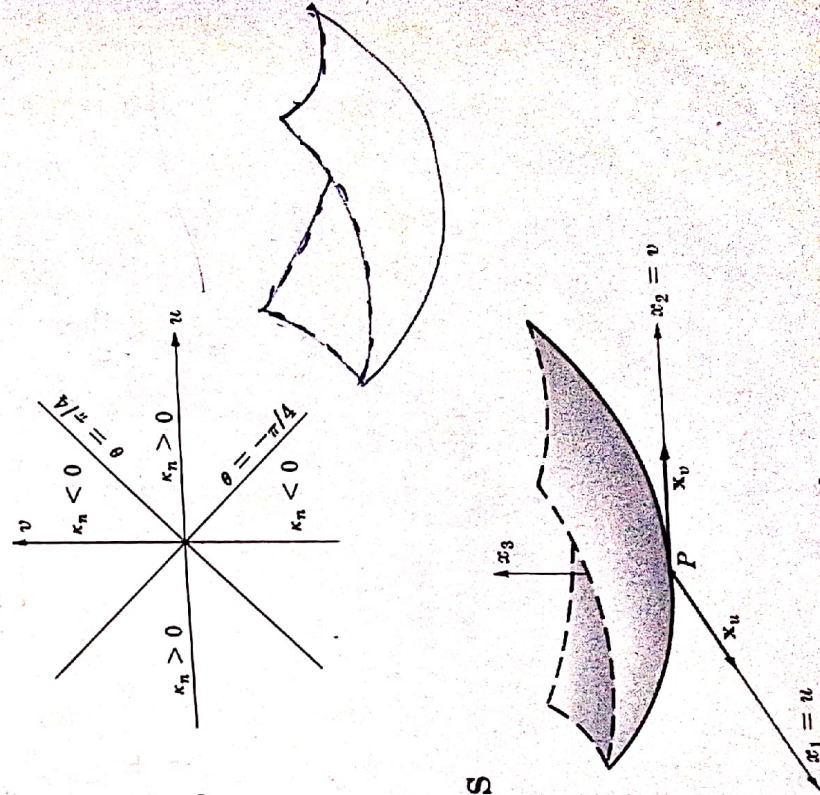


Fig. 9-12

It follows that $E = \mathbf{x}_u \cdot \mathbf{x}_u = 1$, $F = \mathbf{x}_u \cdot \mathbf{x}_v = 0$, $G = \mathbf{x}_v \cdot \mathbf{x}_v = 1$, and

$$\kappa_n = \frac{L du^2 + 2M du dv + N dv^2}{E du^2 + 2F du dv + G dv^2} = \frac{L du^2 + 2M du dv + N dv^2}{du^2 + dv^2}$$

Since κ_n depend only on the ratio du/dv , we can assume that $du^2 + dv^2 = 1$ and set $du = \cos \theta$ and $dv = \sin \theta$. Thus

$$\kappa_n = L \cos^2 \theta + 2M \cos \theta \sin \theta + N \sin^2 \theta$$

Finally we let $|\kappa_n| = 1/r^2$ and set $x_1 = r \cos \theta$, $x_2 = r \sin \theta$, obtaining

$$\pm 1 = Lx_1^2 + 2Mx_1x_2 + Nx_2^2 \quad (9.19)$$

The above equation determines a conic section in the x_1x_2 plane, called *Dupin's indicatrix*, such that the distance r from a point (x_1, x_2) to the origin is the reciprocal of the square root of $|\kappa_n|$ in the direction $\cos \theta : \sin \theta$.

If P is an elliptic point ($LN - M^2 > 0$), the indicatrix is an ellipse as shown in Fig. 9-13(a). If P is a hyperbolic point ($LN - M^2 < 0$), the indicatrix consists of a pair of conjugate hyperbolas (see Fig. 9-13(b)). Along one of the hyperbolas κ_n is positive and along the other κ_n is negative. The common asymptotes correspond to the directions for which $\kappa_n = 0$. In the parabolic case ($LN - M^2 = 0$, $L^2 + N^2 + M^2 \neq 0$) the quantity $Lx_1^2 + 2Mx_1x_2 + Nx_2^2$ factors and the indicatrix is a pair of parallel lines as shown in Fig. 9-13(c), whose directions are also in the direction in which $\kappa_n = 0$. In the planar case ($L = M = N = 0$) the indicatrix does not exist.

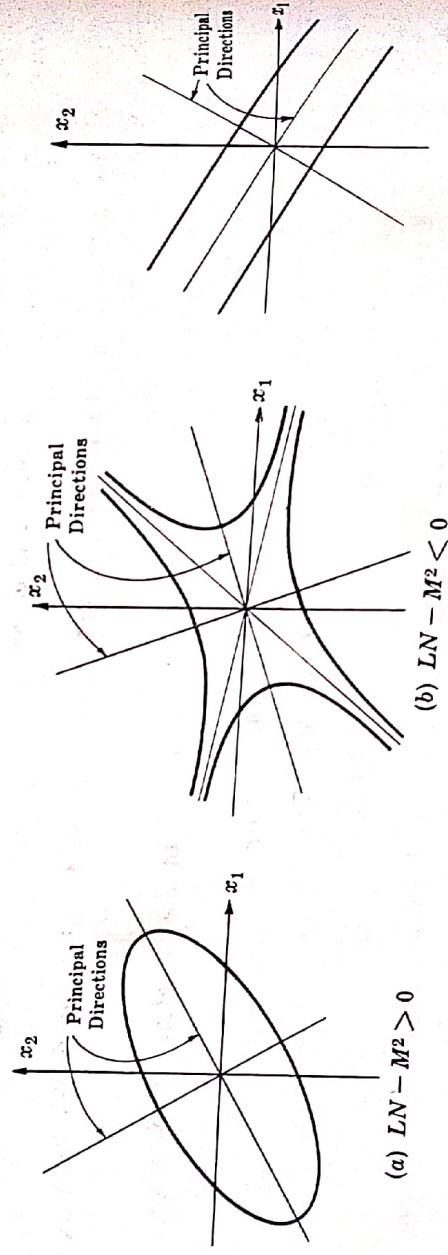


Fig. 9-13

Recalling that $|\kappa_n| = 1/r^2$, we see that if the indicatrix exists and is not a circle, then κ_n takes on distinct maximum and minimum values, say κ_1 and κ_2 , in two *perpendicular* directions, the axes of the indicatrix. At an elliptic point, assuming $\kappa_n > 0$, we see that the maximum κ_1 is taken on in the direction of the minor axis of the indicatrix, i.e. the minimum distance to the origin, and the minimum κ_2 is taken on in the direction of the major axis. At a hyperbolic point the maximum κ_1 is taken on in the direction of the direction of the axis of the hyperbola along which κ_n is positive and is taken on in the parabolic point, assuming $\kappa_n \equiv 0$, the maximum κ_1 is taken on in the direction perpendicular to the parallel lines, and the minimum $\kappa_2 = 0$ is taken on in the direction of the parallel lines of the indicatrix. The two perpendicular directions for which the parallel normal curvatures, κ_1 and κ_2 , are called the *principal directions*, and the corresponding

It remains to consider the elliptic point where the indicatrix is a circle and the planar point where the indicatrix does not exist. At the elliptic point $\kappa_n = \text{constant} \neq 0$ and all directions are called principal directions. Similarly at a planar point $\kappa_n = \text{constant} = 0$ and again all directions are principal directions. A point on the surface at which $\kappa_n = \text{constant}$ is called an *umbilical point*. The elliptic case is called an *elliptic umbilical point*. The planar point is also called a *parabolic umbilical point*.

Example 9.9.

- (a) From Example 9.7, at every point on a sphere of radius a , the normal curvature $\kappa_n = \text{constant} = 1/a$. Every point on a sphere is an elliptic umbilical point and every direction is a principal direction.
- (b) The equation of a plane is $\mathbf{x} = \mathbf{a} + b\mathbf{u} + c\mathbf{v}$, $\mathbf{a}, b, c = \text{constant}$. Here $\mathbf{x}_{uu} = \mathbf{x}_{uv} = \mathbf{x}_{vv} = \mathbf{0}$. Hence $L = M = N = 0$. Every point on a plane is a planar or parabolic umbilical point. Every direction on a plane is a principal direction.

- (c) From Example 9.8, the normal curvature at the origin of $\mathbf{x} = u\mathbf{e}_1 + v\mathbf{e}_2 + (u^2 - v^2)\mathbf{e}_3$

$$\text{is } \kappa_n = \frac{2(du^2 - dv^2)}{du^2 + dv^2}. \quad \text{For } du^2 + dv^2 = 1, \quad du = \cos \theta,$$

$dv = \sin \theta$, $r^2 = 1/|\kappa_n|$, $x_1 = r \cos \theta$, $x_2 = r \sin \theta$, we obtain the indicatrix $\pm 1 = 2(x_1^2 - x_2^2)$. These are conjugate hyperbolas, as shown in Fig. 9-14. Here the maximum value of κ_n is 2 and is taken on in the direction of the x_1 axis. The minimum value of κ_n is -2 and is taken on in the direction of the x_2 axis. The directions of the x_1 and x_2 axes are the principal directions.

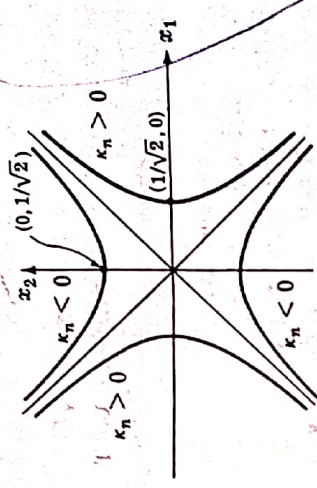


Fig. 9-14

Suppose now $\mathbf{x} = \mathbf{x}(u, v)$ is an arbitrary patch on the surface containing P . In Problem 9.16 we prove

Theorem 9.5. A real number κ is a principal curvature at P in the direction $du:dv$ if and only if κ , du and dv satisfy

$$\begin{aligned} (L - \kappa E) du + (M - \kappa F) dv &= 0 \\ (M - \kappa F) du + (N - \kappa G) dv &= 0 \end{aligned} \quad (9.20)$$

$$\text{where } du^2 + dv^2 \neq 0.$$

The above is a homogeneous system of equations and will have a nontrivial solution du, dv if and only if

$$\det \begin{pmatrix} L - \kappa E & M - \kappa F \\ M - \kappa F & N - \kappa G \end{pmatrix} = 0$$

or, expanding,

$$(EG - F^2)\kappa^2 - (EN + GL - 2FM)\kappa + (LN - M^2) = 0$$

In Problem 9.14 we show that the discriminant of the above equation is greater than or equal to zero. Thus the equation has either two distinct real roots κ_1 and κ_2 , the principal curvatures at a nonumbilical point, or a single real root κ with multiplicity two, the curvature at an umbilical point. Thus we have

Theorem 9.6. A number κ is a principal curvature if and only if κ is a solution of the equation $(EG - F^2)\kappa^2 - (EN + GL - 2FM)\kappa + (LN - M^2) = 0$ (9.21)

every direction is a principal direction. Thus

Finally we note that at an umbilical point every direction is a principal direction. Thus this can only be the case if and only if all coefficients of equations (9.20) vanish. We have

Theorem 9.7. A point is an umbilical point if and only if the fundamental coefficients are proportional, in which case the normal curvature

$$\kappa = \frac{L}{E} = \frac{M}{F} = \frac{N}{G} \quad (9.22)$$

GAUSSIAN AND MEAN CURVATURE

If we divide equation (9.21) by $EG - F^2$, we can write it in the form

$$\kappa^2 - 2H\kappa + K = 0 \quad (9.23)$$

$$H = \frac{1}{2}(\kappa_1 + \kappa_2) = \frac{EN + GL - 2FM}{2(EG - F^2)}$$

where

$$K = \kappa_1\kappa_2 = \frac{LN - M^2}{EG - F^2} \quad (9.24)$$

is the average of the roots κ_1 and κ_2 and is called the *mean curvature* at P and

is the product of the roots and is called the *Gaussian curvature* at P .

Since the normal curvature κ_n of a curve at most changes sign with a change in orientation of the surface, the extreme values of κ_n remain extreme values and, at most, both change sign with a change in orientation (the maximum becoming the minimum, etc.). It follows that the Gaussian curvature $K = \kappa_1\kappa_2$ is an invariant property of the surface, independent of its representation. Note also that $EG - F^2 > 0$. Thus the sign of K agrees with the sign of $LN - M^2$. Namely,

Theorem 9.8. A point on a surface is elliptic if and only if $K > 0$; hyperbolic if and only if $K < 0$; parabolic or planar if and only if $K = 0$.

Example 9.10.

(a) From Example 9.9(a) we see that at each point on a sphere of radius a , the Gaussian curvature $K = \text{constant} = 1/a^2$. The mean curvature is $H = \pm 1/a$ depending on the orientation of the sphere.

(b) From Example 9.9(b) we see that at each point on a plane the Gaussian curvature $K = 0$ and the mean curvature $H = 0$.

(c) Consider the torus

$$\mathbf{x} = (b + a \sin \phi)(\cos \theta)\mathbf{e}_1 + (b + a \sin \phi)(\sin \theta)\mathbf{e}_2 + (a \cos \phi)\mathbf{e}_3$$

From Example 9.5, page 178, $E = (b + a \sin \phi)^2$, $F = 0$, $G = a^2$, $L = (b + a \sin \phi) \sin \phi$, $M = 0$, $N = a$. From equation (9.21), the principal curvatures are the roots of

$$a^2(b + a \sin \phi)^2\kappa^2 - [a(b + a \sin \phi)^2 + a^2(b + a \sin \phi) \sin \phi]\kappa + a(b + a \sin \phi) \sin \phi = 0$$

From the quadratic formula,

$$\kappa = \frac{(b + 2a \sin \phi) \pm b}{2a(b + a \sin \phi)}$$

or $\kappa_1 = \frac{2b + 2a \sin \phi}{2a(b + a \sin \phi)} = \frac{1}{a}$ which is the maximum curvature, and $\kappa_2 = \frac{\sin \phi}{b + a \sin \phi}$ which is the minimum curvature. Observe that κ_1 is the same at every point and is equal to the curvature of the circle generating the torus. The minimum curvature κ_2 varies with ϕ along a meridian. It takes on its maximum $1/(b + a)$ on the outside parallel $\phi = \pi/2$. It is zero on the parallels $\phi = 0$ and $\phi = \pi$, and it takes on its minimum $-1/(b - a)$ on the inside parallel $\phi = -\pi/2$. The Gaussian curvature is $K = \kappa_1\kappa_2 = \frac{\sin \phi}{a(b + a \sin \phi)}$.

LINES OF CURVATURE

Suppose again that P is a point on a patch $\mathbf{x} = \mathbf{x}(u, v)$ on a surface of class ≥ 2 . Problem 9.24, page 197, we prove

Theorem 9.9. A direction $du:dv$ is a principal direction at P if and only if du and dv satisfy

$$(EM - LF) du^2 + (EN - LG) du dv + (FN - MG) dv^2 = 0 \quad (9.25)$$

At a nonumbilical point the above can be shown to factor into two linear equations of the form $A du + B dv = 0$ for the two perpendicular principal directions.

Note that since the normal curvature κ_n of a curve is invariant except for sign, it follows that the directions in which κ_n takes on its extreme value, i.e. the principal directions, are also invariant. In particular if $\mathbf{x}^*(\theta, \phi)$ is any other patch containing P , then $d\theta:d\phi$ is a principal direction if and only if $d\theta = \theta_u du + \theta_v dv$ and $d\phi = \phi_u du + \phi_v dv$, where $du:dv$ is a principal direction at P with respect to $\mathbf{x} = \mathbf{x}(u, v)$.

Now a curve on a surface whose tangent at each point is along a principal direction is called a *line of curvature*. It follows from the above that a curve is a line of curvature if and only if at each point the direction of its tangent satisfies (9.25) for some patch $\mathbf{x} = \mathbf{x}(u, v)$ containing the point. It follows that (9.25) can be regarded as a differential equation for two families of lines of curvature. From an existence and uniqueness theorem for differential equations, solutions to (9.25) exist if the coefficients are of class C^1 . Thus we have

Theorem 9.10. In the neighborhood of a nonumbilical point on a surface of class ≥ 3 there exist two orthogonal families of lines of curvature.

Example 9.11.

Consider the surface

$$\mathbf{x} = ue_1 + ve_2 + (u^2 + v^2)e_3$$

It can be computed that $E = 1 + 4u^2$, $F = 4uv$, $G = 1 + 4v^2$, $L = 2(4u^2 + 4v^2 + 1)^{-1/2}$, $M = 0$, $N = 2(4u^2 + 4v^2 + 1)^{-1/2}$. From (9.25) we obtain, after dividing by $-8(4u^2 + 4v^2 + 1)^{-1/2}$,

$$uv du^2 + (v^2 - u^2) du dv - uv dv^2 = 0$$

or $(u du + v dv)(v du - u dv) = 0$, or

$$u du + v dv = 0, \quad v du - u dv = 0$$

The solutions to the first equation is the family of circles $u^2 + v^2 = r^2$, and the solutions to the second equation is the family of lines through the origin, $u = bv$. The images of those curves on the surface are the lines of curvature, as shown in Fig. 9-15. Note that at the origin $u = 0, v = 0$, we have $E = 1$, $F = 0$, $G = 1$, $L = 2$, $M = 0$, $N = 2$. Namely, the first and second fundamental coefficients are proportional and we have an umbilical point, with every direction a principal direction.

As a consequence of Theorem 9.10, if a surface is sufficiently smooth, one can introduce a coordinate patch of class C^2 in the neighborhood of a nonumbilical point P such that the u and v parameter curves are themselves the lines of curvature. In many problems however it is sufficient to have a coordinate patch of class C^2 containing P such that the u and v parameter curves are in the direction of the principal direction simply at P . In Problem 9.21 we prove

Theorem 9.11. For every point P on a surface of class ≥ 2 there exists a coordinate patch containing P such that the directions of the u and v parameter curves at P are principal directions.

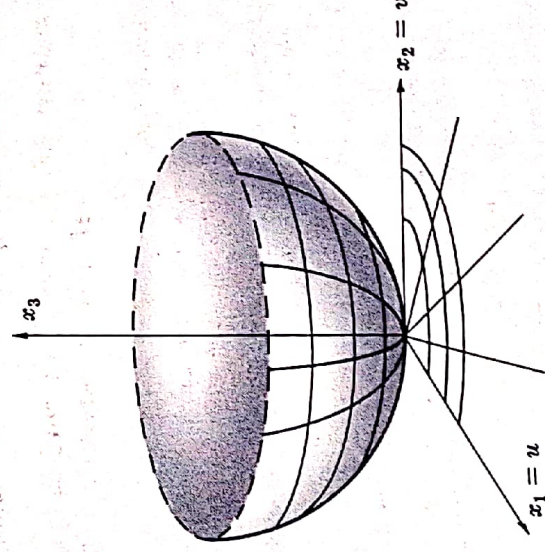


Fig. 9-15

Now suppose that the directions of the u - and v -parameter curves at a nonumbilical point P on a patch are principal directions. Since $\mathbf{x}_u = \mathbf{l}\mathbf{x}_u + 0\mathbf{x}_v$ is tangent to the u -parameter curves and $\mathbf{x}_v = 0\mathbf{x}_u + \mathbf{l}\mathbf{x}_v$ is tangent to the v -parameter curves, it follows that at P equation (9.25) must be satisfied for $du = 1$, $dv = 0$ and $dv = 1$, $du = 0$. Substituting, we find that $FN - MG = 0$ and $EM - LF = 0$.

We recall further that the principal directions at a nonumbilical point are orthogonal. Hence from Theorem 9.1, page 173, $F = 0$ at P and so $MG = EM = 0$. Finally, since $\mathbf{l}$ is positive definite, $E > 0$ and so $M = 0$. Thus $F = M = 0$ at P . The converse is also true. Namely,

Theorem 9.12. The directions of the u - and v -parameter curves at a nonumbilical point on a patch are in the direction of the principal directions if and only if $F = M = 0$ at the point.

As a consequence of the above we also have

Corollary: The u - and v -parameter curves on a patch without umbilical points are lines of curvature if and only if at every point on the patch $F = M = 0$.

If the directions of the u - and v -parameter curves at a point P on a patch are principal directions, we obtain simple expressions for the principal curvatures. Namely, suppose first that P is a nonumbilical point. Then, from Theorem 9.12 above, $F = M = 0$ at P and equations (9.20) for the principal curvatures and directions at P reduce to $(L - \kappa E)du = 0$ and $(N - \kappa G)dv = 0$. Since $du = 1$, $dv = 0$ are the direction numbers for the u -parameter curve through P , it follows that the principal curvature in this direction is $\kappa_1 = L/E$. The principal curvature in the direction of the v -parameter curve is $\kappa_2 = N/G$ and is obtained by using $du = 0$ and $dv = 1$. If P is an umbilical point then in any case from Theorem 9.7, $\kappa = L/E = M/F = N/G$. Thus we have proved

Theorem 9.13. If the directions of the u - and v -parameter curves at a point P on a patch are principal directions, then the principal curvatures at P are given by

$$\kappa_1 = L/E \quad \text{and} \quad \kappa_2 = N/G$$

Corollary: If the u - and v -parameter curves on a patch are lines of curvature, then at each point the principal curvatures are given by

$$\kappa_1 = L/E \quad \text{and} \quad \kappa_2 = N/G$$

Example 9.12.

Consider the surface

$$\mathbf{x} = u\mathbf{e}_1 + v\mathbf{e}_2 + (u^2 + v^2)\mathbf{e}_3$$

From Example 9.11 the lines of curvature are the images of the circles $u^2 + v^2 = r^2$ and lines $u = br$. The lines of curvature can be introduced as parameter lines by the parameter transformation $u = r \cos \theta$, $v = r \sin \theta$. This gives the representation

$$\mathbf{x} = (r \cos \theta)\mathbf{e}_1 + (r \sin \theta)\mathbf{e}_2 + r^2\mathbf{e}_3, \quad r > 0$$

It can be computed that here the first and second fundamental coefficients have the simpler form $E = r^2$, $F = 0$, $G = 1 + 4r^2$, $L = -2r^2(1 + 4r^2)^{-1/2}$, $M = 0$, $N = -2(1 + 4r^2)^{-1/2}$ and the principal curvatures are $\kappa_1 = L/E = -2(1 + 4r^2)^{-1/2}$ and $\kappa_2 = N/G = -2(1 + 4r^2)^{-3/2}$.

RODRIGUES' FORMULA

Suppose $du : dv$ is a principal direction at a point P on a patch and κ the corresponding principal curvature. Then from equations (9.10), (9.12) and (9.20) we obtain

$$\begin{aligned} (-N_u \cdot \mathbf{x}_u - \kappa \mathbf{x}_u \cdot \mathbf{x}_u) du + (-N_v \cdot \mathbf{x}_u - \kappa \mathbf{x}_v \cdot \mathbf{x}_u) dv &= 0 \\ (-N_u \cdot \mathbf{x}_v - \kappa \mathbf{x}_u \cdot \mathbf{x}_v) du + (-N_v \cdot \mathbf{x}_v - \kappa \mathbf{x}_v \cdot \mathbf{x}_v) dv &= 0 \end{aligned}$$

$$[(N_u du + N_v dv) + \kappa(x_u du + x_v dv)] \cdot x_u = 0$$

$$[(N_u du + N_v dv) + \kappa(x_u du + x_v dv)] \cdot x_v = 0$$

$$(dN + \kappa dx) \cdot x_u = 0, \quad (dN + \kappa dx) \cdot x_v = 0$$

of But $dN + \kappa dx$ is parallel to the tangent plane at P since dN and dx are; and the vectors x_u and x_v are independent. Hence $dN + \kappa dx = 0$ or $dN = -\kappa dx$. Thus, in the direction of a principal direction, the vector dN is parallel to dx and is given by $dN = -\kappa dx$, where κ is the principal curvature in that direction. The converse of this is proved in Problem 9.23, page 196. Thus

Theorem 9.14. The direction $du:dv$ is a principal direction at a point on a patch if and only if for some scalar κ , $dN = N_u du + N_v dv$ and $dx = x_u du + x_v dv$ satisfy

$$dN = -\kappa dx \quad (9.26)$$

When such is the case κ is the principal curvature in the direction $du:dv$.

The above formula, which completely characterizes the principal directions, is called *Rodrigues' formula*, and should be committed to memory.

ASYMPTOTIC LINES — CONJUGATE FAMILIES OF CURVES

A direction at a point on a patch for which

$$II = L du^2 + 2M du dv + N dv^2 = 0 \quad (9.27)$$

is called an *asymptotic direction*. Since $\kappa_n = II/I$ and I is positive definite, the asymptotic directions are also the directions in which $\kappa_n = 0$. At an elliptic point there are no asymptotic directions; at a hyperbolic point there are two distinct asymptotic directions; at a parabolic point there is one asymptotic direction; and at a planar point every direction is asymptotic.

A curve on a surface which is tangent to an asymptotic direction at every point is called an *asymptotic line*. Thus a curve on a surface is an asymptotic line if and only if the direction of the tangent line to the curve at each point satisfies (9.27) for some patch $x = x(u, v)$ containing the point. At a hyperbolic point this equation has two real distinct factors of the form $A du + B dv = 0$, which can be regarded as first order differential equations for the asymptotic lines. As in the case of the lines of curvature, we have

Theorem 9.15. In a neighborhood of a hyperbolic point on a surface of class ≥ 3 there exist two distinct families of asymptotic lines.

If the u - and v -parameter curves on a patch are themselves asymptotic lines, then at each point (9.27) must be satisfied for $du = 1, dv = 0$ and $du = 0, dv = 1$. Hence $L = N = 0$. The converse is also true. Namely,

Theorem 9.16. The u - and v -parameter curves on a patch are asymptotic lines if and only if at each point $L = N = 0$.

Example 9.13.

Consider the surface

$$x = (r \cos \theta)e_1 + (r \sin \theta)e_2 + (\log r)e_3, \quad r > 0$$

It can be computed that

$$L = -r/(1 + r^2)^{1/2}, \quad M = 0, \quad N = 1/r(1 + r^2)^{1/2}$$

Multiplying (9.27) by $(1+r^2)^{1/2}$ gives the differential equation $-rd\theta^2 + dr^2/r = 0$ or $d\theta + dr/r = 0$ or $d\theta - dr/r = 0$. These have solutions $\theta + u = \log r$ and $\theta + v = -\log r$, where u and v are the constants of integration. The images of these curves on the surface are two families of asymptotic lines. Solve for θ and r in terms of u and v gives $\theta = -(u+v)/2$, $r = e^{(u-v)/2}$. Then

$$\mathbf{x} = e^{(u-v)/2}(\cos \frac{1}{2}(u+v))\mathbf{e}_1 + e^{(u-v)/2}(\sin \frac{1}{2}(u+v))\mathbf{e}_2 + \frac{1}{2}(u-v)\mathbf{e}_3$$

which is a parameterization of the surface by asymptotic lines.

Now we recall that at each point on a curve on a patch the normal curvature

$$\kappa_n = \text{II/I} = \mathbf{k} \cdot \mathbf{N}$$

where $\mathbf{k}$ is the curvature vector of the curve. It follows that a curve is an asymptotic line if and only if at each point $\mathbf{k} \cdot \mathbf{N} = 0$; i.e. if and only if $\mathbf{k} = 0$, or $\mathbf{k}$ is perpendicular to $\mathbf{N}$. Thus we have

Theorem 9.17. A curve on a surface is an asymptotic line if and only if every point on the curve is either a point of inflection ($\mathbf{k} = 0$) or the osculating plane at the point is tangent to the surface.

Since along a straight line $\mathbf{k} \equiv 0$, we have

Corollary: A straight line on a surface is an asymptotic line.

For asymptotic lines which are not straight lines we will prove in Problem 9.30, page 199,

Theorem 9.18. (Beltrami-Enneper). At each point on an asymptotic line which is not a straight line the torsion satisfies

$$\tau^2 = -K$$

where K is the Gaussian curvature at the point.

A direction $\delta u : \delta v$ at a point on a patch is said to be *conjugate* to a direction $du : dv$ if

$$d\mathbf{x} \cdot \delta \mathbf{N} = 0 \quad (9.28)$$

where $d\mathbf{x} = \mathbf{x}_u du + \mathbf{x}_v dv$ and $\delta \mathbf{N} = \mathbf{N}_u \delta u + \mathbf{N}_v \delta v$. Expanding and using (9.10), we find that the above is equivalent to

$$L du \delta u + M(du \delta v + dv \delta u) + N dv \delta v = 0 \quad (9.29)$$

Note. Because of symmetry, it follows that $\delta u : \delta v$ is also conjugate to $du : dv$, so that we can speak of the conjugate directions $du : dv$ and $\delta u : \delta v$. Also note that an *asymptotic direction* is *self-conjugate*.

Given an arbitrary direction $du : dv$, (9.29) is a linear equation

$$(L du + M dv) \delta u + (M du + N dv) \delta v = 0$$

for $\delta u : \delta v$. In Problem 9.29 we will prove that this equation has a unique solution $\delta u : \delta v$ if $LN - M^2 \neq 0$. Namely,

Theorem 9.19. At an elliptic or hyperbolic point on a surface every direction has a unique conjugate direction.

Two families of curves on a surface are said to be *conjugate families* of curves if the direction of their tangents are conjugate at each point. Given a one parameter family of curves $f(u, v) = C_1$ in the parameter plane, equation (9.29) can be regarded as a first order differential equation for a conjugate family $g(u, v) = C_2$.

If the u - and v -parameter curves on a patch are themselves conjugate families of curves, then (9.29) must be satisfied at each point by $du = 1$, $dv = 0$ and $\delta u = 0$, $\delta v = 1$. Substituting into (9.29) gives $M = 0$. The converse is also true. That is, we have

Theorem 9.20. The u - and v -parameter curves on a patch are conjugate families of curves if and only if at each point $M = 0$.

Note, as a consequence of the above theorem and the corollary to Theorem 9.12 we have

Corollary: The u - and v -parameter curves on a patch without umbilical points are orthogonal and conjugate families of curves if and only if they are lines of curvature.

Example 9.14.

Consider the surface (see Example 9.11)

$$\mathbf{x} = u\mathbf{e}_1 + v\mathbf{e}_2 + (u^2 + v^2)\mathbf{e}_3$$

Here $L = N = 2(u^2 + v^2 + 1)^{-1/2}$, $M = 0$. Observe that the surface is already represented by conjugate families of parameter curves, since $M = 0$ everywhere. Now consider the family of curves in the parameter plane given by $f(u, v) = u^2 + v^2 = C_1^2$. Since $df = f_u du + f_v dv = 2u du + 2v dv = 0$ we have $du : dv = f_v : -f_u = -v : u$. Using (9.29) and dividing by $2(1 + u^2 + v^2)^{-1/2}$ gives $-v \delta u + u \delta v = 0$, which has as solutions the family of lines $u = C_2 v$. The two families of curves $u^2 + v^2 = C_1^2$ and $u = C_2 v$ define two conjugate families of curves on the surface which are in fact the lines of curvature.

Solved Problems

FIRST FUNDAMENTAL FORM. ARC LENGTH. SURFACE AREA

9.1. Show that the first fundamental form on the surface of revolution

$$\mathbf{x} = f(t)(\cos \theta)\mathbf{e}_1 + f(t)(\sin \theta)\mathbf{e}_2 + g(t)\mathbf{e}_3$$

$$I = f^2 d\theta^2 + (f'^2 + g'^2) dt^2$$

is

$$\mathbf{x}_\theta = -(f \sin \theta)\mathbf{e}_1 + (f \cos \theta)\mathbf{e}_2, \quad \mathbf{x}_t = (f' \cos \theta)\mathbf{e}_1 + (f' \sin \theta)\mathbf{e}_2 + g'\mathbf{e}_3$$

$$E = \mathbf{x}_\theta \cdot \mathbf{x}_\theta = f^2, \quad F = \mathbf{x}_\theta \cdot \mathbf{x}_t = 0, \quad G = \mathbf{x}_t \cdot \mathbf{x}_t = f'^2 + g'^2$$

from which the result follows.

9.2. Find the length of the arc $u = e^{\theta(\cot \beta)/\sqrt{2}}$, $\theta = 0$, $0 \leq \theta \leq \pi$, $\beta = \text{constant}$, on the cone

$$\mathbf{x} = (u \cos \theta)\mathbf{e}_1 + (u \sin \theta)\mathbf{e}_2 + u\mathbf{e}_3$$

$$G = \mathbf{x}_u \cdot \mathbf{x}_u = 2, \quad du/d\theta = u(\cot \beta)/\sqrt{2}, \quad d\theta/d\theta = 1.$$

$$E = \mathbf{x}_\theta \cdot \mathbf{x}_\theta = u^2, \quad F = \mathbf{x}_\theta \cdot \mathbf{x}_u = 0, \quad S = \int_0^\pi \left[E \left(\frac{d\theta}{d\theta} \right)^2 + 2F \frac{du}{d\theta} \frac{d\theta}{d\theta} + G \left(\frac{du}{d\theta} \right)^2 \right]^{1/2} d\theta$$

$$= \int_0^\pi [u^2 + (\cot^2 \beta) u^2]^{1/2} d\theta = \sqrt{1 + \cot^2 \beta} \int_0^\pi u d\theta$$

$$= \int_0^\pi [u^2 + (\cot^2 \beta) u^2]^{1/2} d\theta = \frac{\sqrt{2}}{\cos \beta} \int_0^\pi (e^{\pi(\cot \beta)/\sqrt{2}} - 1) d\theta$$

$$= \sqrt{1 + \cot^2 \beta} \int_0^\pi e^{\theta(\cot \beta)/\sqrt{2}} d\theta = \frac{\sqrt{2}}{\cos \beta} (e^{\pi(\cot \beta)/\sqrt{2}} - 1)$$

which is the required result.

- 9.3. Show that the curve in Problem 9.2 above intersects the generating lines $\theta = \text{constant}$ on the cone at a constant angle β .

$$\begin{aligned} \cos \angle \left(\frac{dx}{d\theta}, \mathbf{x}_u \right) &= \frac{\left(\frac{dx}{d\theta} \cdot \mathbf{x}_u \right)}{\left| \frac{dx}{d\theta} \right| |\mathbf{x}_u|} = \frac{(\mathbf{x}_0(d\theta/d\theta) + \mathbf{x}_u(du/d\theta)) \cdot \mathbf{x}_u}{|\mathbf{x}_0(d\theta/d\theta) + \mathbf{x}_u(du/d\theta)| |\mathbf{x}_u|} = \frac{F + G(du/d\theta)}{\sqrt{E + 2F(du/d\theta) + G(du/d\theta)^2} \sqrt{G}} \\ &= \frac{\sqrt{2}(du/d\theta)}{\sqrt{u^2 + 2(du/d\theta)^2}} = \frac{u \cot \beta}{\sqrt{u^2 + u^2 \cot^2 \beta}} = \frac{\cot \beta}{\sqrt{1 + \cot^2 \beta}} = \cos \beta \end{aligned}$$

which is the required result.

- 9.4. If the first fundamental form on a patch is of the form $I = du^2 + f(u, v) dv^2$, prove that the v -parameter curves cut off equal segments from all u -parameter curves. Note, since $F = 0$, the parameter curves are also orthogonal. In this case the v -parameter curves are said to be *parallel*, as shown in Fig. 9-16.

The distance along a u -parameter curve $u = u$, $v = v_0$ between $u = u_1$ and $u = u_2$, $u_1 < u_2$, is

$$d = \int_{u_1}^{u_2} [(du/du)^2 + f(u, v_0)(dv/du)^2]^{1/2} du$$

Since $du/du = 1$ and $dv/du = 0$, we have

$$d = \int_{u_1}^{u_2} du = u_2 - u_1$$

which shows that d is the same for each $v = v_0$.

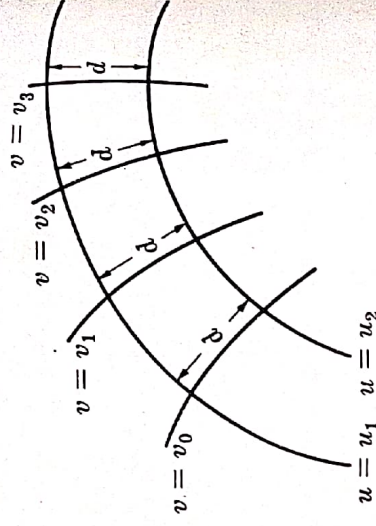


Fig. 9-16

- 9.5.

Show that for a surface represented by $f(x_1, x_2, x_3) = C$, we have $f_{x_1} dx_1 + f_{x_2} dx_2 + f_{x_3} dx_3 = 0$ where $dx = (dx_1, dx_2, dx_3)$ is an arbitrary tangent vector to the surface at $P(x_1, x_2, x_3)$.

Suppose $\mathbf{x} = x_1(t)\mathbf{e}_1 + x_2(t)\mathbf{e}_2 + x_3(t)\mathbf{e}_3$ is a regular curve on the surface through $P(x_1, x_2, x_3)$. Then for all t , $f(x_1(t), x_2(t), x_3(t)) = C$. Hence $df/dt = f_{x_1}(dx_1/dt) + f_{x_2}(dx_2/dt) + f_{x_3}(dx_3/dt) = 0$. Namely, if $dx = dx_1\mathbf{e}_1 + dx_2\mathbf{e}_2 + dx_3\mathbf{e}_3$ is an arbitrary vector in the tangent plane at P , then $f_{x_1} dx_1 + f_{x_2} dx_2 + f_{x_3} dx_3 = 0$.

- 9.6.

Show that the parameter transformation $u = \sinh t$, $\phi = \theta$, $-\infty < t < \infty$, $0 < \theta < 2\pi$, between the points in the parameter plane of the surface of revolution M_1 ,

$$\mathbf{x} = (\cosh t \cos \theta)\mathbf{e}_1 + (\cosh t \sin \theta)\mathbf{e}_2 + t\mathbf{e}_3, \quad 0 < \theta < 2\pi$$

and the points in the parameter plane of the right conoid M_2 ,

$$\mathbf{y} = (u \cos \phi)\mathbf{e}_1 + (u \sin \phi)\mathbf{e}_2 + \phi\mathbf{e}_3, \quad 0 < \phi < 2\pi$$

determines a 1-1 correspondence between the points on the surfaces themselves such that the first fundamental forms agree on corresponding vectors.

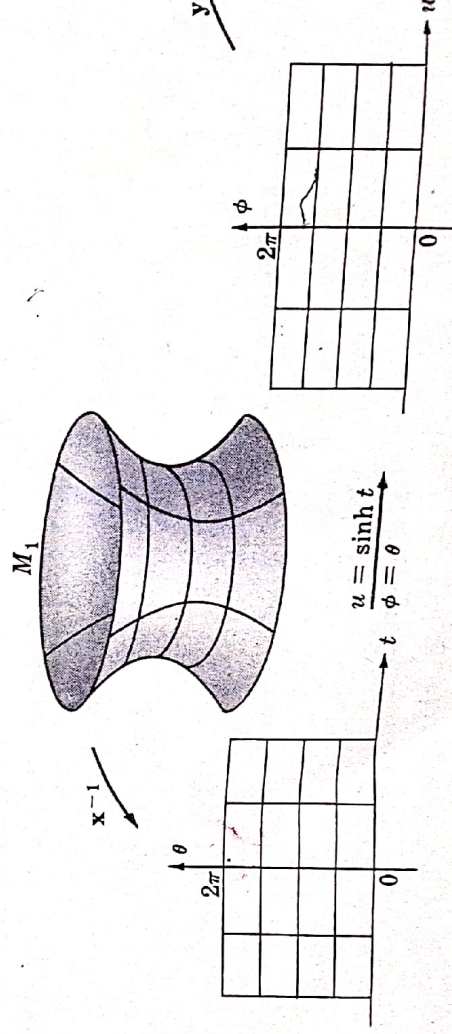


Fig. 9-17

Since $\mathbf{x}$ and y are both 1-1 mappings of their corresponding domains onto the surfaces M_1 and M_2 and $u = \sinh t$, $\phi = \theta$, defines a 1-1 correspondence between the domains, it follows that the composite mapping $\mathbf{x}^{-1}$ followed by $u = \sinh t$, $\phi = \theta$, followed by y is a 1-1 mapping of the surface M_1 onto M_2 .

Now, at the point $\mathbf{x}(t, \theta)$ on M_1 we have $\mathbf{x}_t = \mathbf{x}_\theta \cdot \mathbf{x}_\theta = \cosh^2 t$, $\mathbf{F} = \mathbf{x}_\theta \cdot \mathbf{x}_t = 0$, $G = \mathbf{x}_t \cdot \mathbf{x}_t = \cosh^2 t$ and $I = \cosh^2 t (d\theta^2 + dt^2)$. At $y(u, \phi)$ on M_2 , $\mathbf{F}^* = y_\phi \cdot y_\phi = u^2 + 1$, $\mathbf{F}^* = y_\phi \cdot y_u = 0$, $G^* = y_u \cdot y_u = 1$ and $I^* = (u^2 + 1) d\phi^2 + du^2$, but $u = \sinh t$, $\phi = \theta$, $d\phi = d\theta$, $du = \cosh t dt$. Hence $I^* = (\sinh^2 t + 1) d\theta^2 + \cosh^2 t dt^2 = \cosh^2 t (d\theta^2 + dt^2) = I$ which is the required result.

SECOND FUNDAMENTAL FORM

97. Show that the surface

$$\mathbf{x} = u\mathbf{e}_1 + v\mathbf{e}_2 + (u^2 + v^2)\mathbf{e}_3$$

is elliptic where $v > 0$, hyperbolic where $v < 0$, and parabolic for $v = 0$.

$$\mathbf{x}_u = \mathbf{e}_1 + 2u\mathbf{e}_3, \quad \mathbf{x}_v = \mathbf{e}_2 + 2v\mathbf{e}_3$$

$$\mathbf{N} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{|\mathbf{x}_u \times \mathbf{x}_v|} = \frac{(4u^2 + 9v^4 + 1)^{-1/2} (-2u\mathbf{e}_1 - 3v^2\mathbf{e}_2 + \mathbf{e}_3)}{(4u^2 + 9v^4 + 1)^{-1/2}}$$

$$\mathbf{x}_{uu} = 2\mathbf{e}_3, \quad \mathbf{x}_{uv} = 0, \quad \mathbf{x}_{vv} = 6v\mathbf{e}_3$$

$$L = \mathbf{x}_{uu} \cdot \mathbf{N} = 2(4u^2 + 9v^4 + 1)^{-1/2}, \quad M = \mathbf{x}_{uv} \cdot \mathbf{N} = 0$$

$$N = \mathbf{x}_{vv} \cdot \mathbf{N} = 6v(4u^2 + 9v^4 + 1)^{-1/2}, \quad LN - M^2 = \frac{12v}{(4u^2 + 9v^4 + 1)}$$

Since $(4u^2 + 9v^4 + 1) > 0$ for all (u, v) , we have $LN - M^2 > 0$ for $v > 0$, $LN - M^2 < 0$ where $v < 0$, and $LN - M^2 = 0$ where $v = 0$. Thus the surface is elliptic for $v > 0$, hyperbolic for $v < 0$, and since $L \neq 0$ for all (u, v) , parabolic for $v = 0$.

98. Show that the surface in Problem 9.7 lies on both sides of the tangent plane in every neighborhood of the parabolic point $P(0, 0)$.

At $u = 0$, $v = 0$ we have $\mathbf{x}_u = \mathbf{e}_1$ and $\mathbf{x}_v = \mathbf{e}_2$; hence the tangent plane at $P(0, 0)$ is the coordinate plane $x_3 = 0$. The v -parameter curve $u = 0$ is the cubic $\mathbf{x} = v\mathbf{e}_2 + v^3\mathbf{e}_3$ which lies on both sides of the tangent plane in every neighborhood of $P(0, 0)$ as shown in Fig. 9-18.

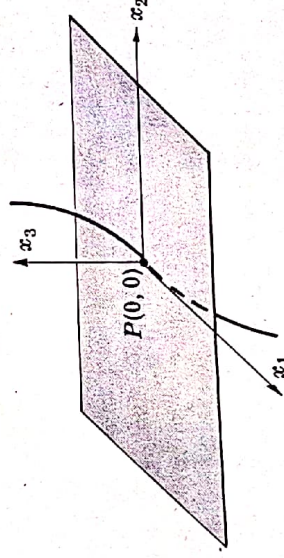


Fig. 9-18

99. Prove that all points on the tangent surface of a curve are parabolic.

The tangent surface to the curve $y = y(s)$ is $\mathbf{x} = y(s) + ut(s)$. Here

$$\mathbf{x}_s = \dot{y} + u\dot{t} = t + u\kappa\mathbf{n}, \quad \mathbf{x}_u = t$$

$$\mathbf{N} = \frac{\mathbf{x}_u \times \mathbf{x}_s}{|\mathbf{x}_u \times \mathbf{x}_s|} = \frac{u\kappa\mathbf{h}}{|\kappa u|}, \quad \mathbf{x}_{ss} = \dot{t} + u\dot{\kappa}\mathbf{n} + u\kappa\dot{\mathbf{n}} = -u\kappa^2\mathbf{t} + (u\dot{\kappa} + \kappa)\mathbf{n} + \frac{u\kappa\tau\mathbf{b}}{|\kappa u|}$$

$$M = \mathbf{x}_{su} \cdot \mathbf{N} = \frac{|\kappa u| \tau}{|\kappa u|}, \quad L = \mathbf{x}_{ss} \cdot \mathbf{N} = \frac{u\kappa^2}{|\kappa u|}, \quad N = \mathbf{x}_{uu} \cdot \mathbf{N} = 0$$

Hence at all points $LN - M^2 = 0$, which is the required result.

100. Show that every point on the surface of revolution

$$f(t) > 0$$

$\mathbf{x} = f(t)(\cos \theta)\mathbf{e}_1 + f(t)(\sin \theta)\mathbf{e}_2 + t\mathbf{e}_3$, is a parabolic point if and only if the surface is a right circular cylinder $f = a$, or a cone $f = at + b$, $a = \text{constant} \neq 0$, $b = \text{constant}$.

$$L \quad M \quad N$$

It can be computed that $L = \frac{-f}{[1+f'^2]^{1/2}}$, $M = 0$, $N = \frac{f''}{[1+f'^2]^{1/2}}$. Hence

$$LN - M^2 = \frac{-ff''}{1+f'^2}$$

Since $f > 0$, $LN - M^2 = 0$ if and only if $f'' = 0$. That is, if and only if $f = a \neq 0$ or $f = a(x + y)$, $a \neq 0$, which completes the proof.

9.11. Prove that a surface lies on one side of the tangent plane in a neighborhood of an elliptic point.

* Let $P = \mathbf{x}(u, v)$ be an elliptic point, and $Q = \mathbf{x}(u + du, v + dv)$ a neighboring point. We recall that $d = \mathbf{PQ} \cdot \mathbf{N} = \frac{1}{2} \text{II} + o(du^2 + dv^2)$, where $\frac{1}{2} \text{II} = \frac{1}{2}(L du^2 + 2M du dv + N dv^2)$ is an elliptic paraboloid in (du, dv) maintaining the same sign for all (du, dv) and equal to zero if and only if $du = 0$, $dv = 0$. Without loss of generality we can assume $\frac{1}{2} \text{II} > 0$. Now let $du = r \cos \theta$, $dv = r \sin \theta$ and consider the quantity

$$\frac{1}{2} \frac{\text{II}}{du^2 + dv^2} = \frac{1}{2} \frac{L du^2 + 2M du dv + N dv^2}{du^2 + dv^2} = \frac{1}{2} (L \cos^2 \theta + 2M \cos \theta \sin \theta + N \sin^2 \theta)$$

But this is just $\frac{1}{2} \text{II}$ evaluated on the unit circle $du = \cos \theta$, $dv = \sin \theta$. Since $\frac{1}{2} \text{II}$ is continuous and greater than zero on the circle, $\frac{1}{2} \text{II}/(du^2 + dv^2)$ takes on a minimum $m > 0$. Now select ϵ such that $\frac{o(du^2 + dv^2)}{du^2 + dv^2} < m$ for $du^2 + dv^2 < \epsilon^2$. But then

$$\frac{d}{du^2 + dv^2} = \frac{1}{2} \frac{\text{II}}{du^2 + dv^2} + \frac{o(du^2 + dv^2)}{du^2 + dv^2} > 0 \quad \text{for } 0 < du^2 + dv^2 < \epsilon^2$$

It follows that $d > 0$ or Q lies on the same side of the tangent plane for $du^2 + dv^2 < \epsilon^2$, which completes the proof.

NORMAL CURVATURE. GAUSSIAN AND MEAN CURVATURE

9.12. Find the normal curvature vector $\mathbf{k}_n$ and normal curvature κ_n of the curve $u = t$, $v = t$ on the surface $\mathbf{x} = u\mathbf{e}_1 + v\mathbf{e}_2 + (u^2 + v^2)\mathbf{e}_3$ at the point $t = 1$.

$$E = 1 + 4u^2, \quad F = 4uv, \quad G = 1 + 4v^2, \quad N = (4u^2 + 4v^2 + 1)^{-1/2}(-2ue_1 - 2ve_2 + e_3), \quad L = 2(4u^2 + 4v^2 + 1)^{-1/2}, \quad M = 0, \quad N = 2(4u^2 + 4v^2 + 1)^{-1/2}, \quad du/dt = 2t, \quad dv/dt = 1.$$

At $t = 1$: $u = 1$, $v = 1$, $E = 5$, $F = 4$, $G = 5$, $N = -1/3[2e_1 + 2e_2 - e_3]$, $L = 2/3$, $M = 0$, $N = 2/3$, $du/dt = 2$, $dv/dt = 1$. Hence

$$\mathbf{k}_n = \frac{L(du/dt)^2 + 2M(du/dt)(dv/dt) + N(dv/dt)^2}{E(du/dt)^2 + 2F(du/dt)(dv/dt) + G(dv/dt)^2} = \frac{10}{123}$$

and $\mathbf{k}_n = \kappa_n \mathbf{N} = -\frac{10}{369}[2e_1 + 2e_2 - e_3]$, which are the required results.

9.13. Let L be a tangent line to a surface at a point P in a direction in which $\kappa_n \neq 0$ (i.e. a nonasymptotic direction). Prove that the osculating circles of all curves through P tangent to L lie on a sphere.

Let C be a curve through P tangent to L . Since $\mathbf{k}_n = \mathbf{k} \cdot \mathbf{N} \neq 0$, it follows that $\mathbf{k} \neq 0$ and we can write $\mathbf{k}_n = \kappa(\mathbf{n} \cdot \mathbf{N}) = \kappa \cos \alpha$ where, by selecting $\mathbf{N}$ so that $\kappa_n > 0$ and $\mathbf{n}$ in the direction of $\mathbf{k}$, we have $\kappa_n > 0$, $\kappa > 0$, and $0 \leq \alpha = \angle(\mathbf{n}, \mathbf{N}) \leq \pi/2$. Now let $\rho = 1/\kappa$, $R = 1/\kappa_n$; then $\rho = R \cos \alpha$ where $R = \text{constant}$ and ρ is the radius of curvature of the osculating circle to C . It follows that the osculating circle is the intersection of the osculating plane and the sphere of radius R tangent to the tangent plane at P as shown in Fig. 9-19, which is the required result.

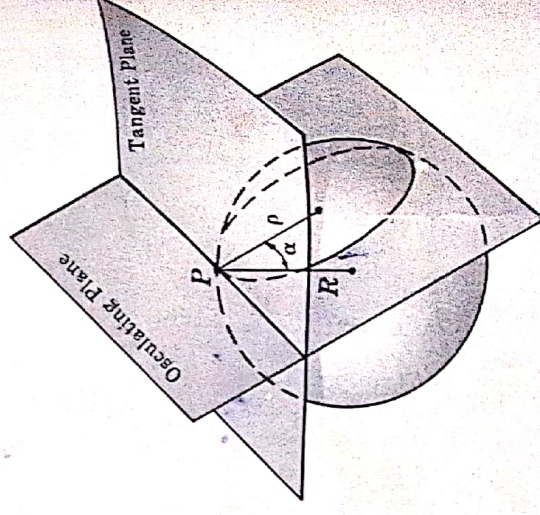


Fig. 9-19

914. Prove that the discriminant of the equation

$$(EG - F^2)\kappa^2 - (EN + GL - 2FM)\kappa + (LN - M^2) = 0$$

is greater than or equal to zero, and equal to zero if and only if $L/E' = M/F = N/G$. The discriminant is $(EN + GL - 2FM)^2 - 4(EG - F^2)(LN - M^2)$, which can be shown to be identically equal to

$$4\left(\frac{EG - F^2}{E^2}\right)(EM - FL)^2 + \left[EN - GL - \frac{2F}{E}(EM - FL)\right]^2$$

Hence the discriminant is greater than or equal to zero. Since $EG - F^2 > 0$, the above is zero if and only if $EM - FL = 0$ and $EN - GL - \frac{2F}{E}(EM - FL) = 0$, or if and only if $EM - FL = 0$ and $EN - GL = 0$ or if and only if $L/E = M/F = N/G$.

915. Prove that at every point P on a surface there is a paraboloid such that the normal curvature of the surface at P in any direction is the same as the paraboloid.

Suppose the surface is translated and rotated so that P is at the origin and the tangent plane is the x_1x_2 plane. Then a neighborhood of P can be represented by

$$\mathbf{x} = ue_1 + ve_2 + f(u, v)e_3$$

where $\mathbf{x}(0, 0) = 0$, $\mathbf{x}_u(0, 0) = \mathbf{e}_1$, $\mathbf{x}_v(0, 0) = \mathbf{e}_2$. From Taylor's theorem,

$$\mathbf{x} = ue_1 + ve_2 + \frac{1}{2}(au^2 + 2buv + cv^2)e_3 + o(u^2 + v^2)$$

where $\mathbf{x}_{uu}(0, 0) = ae_3$, $\mathbf{x}_{uv}(0, 0) = be_3$ and $\mathbf{x}_{vv}(0, 0) = ce_3$, $N(0, 0) = e_3$, and hence $\kappa_n =$

$$a \frac{du^2 + 2b \, du \, dv + c \, dv^2}{du^2 + dv^2}. \quad \text{The surface represented by the approximation}$$

$$\mathbf{x}^* = ue_1 + ve_2 + \frac{1}{2}(au^2 + 2buv + cv^2)e_3$$

is a paraboloid tangent to the x_1x_2 plane at $u = 0$, $v = 0$ and such that $\kappa_n^* = \kappa_n$, which is the required result.

916. Prove Theorem 9.5. That is, prove that κ_0 is a principal curvature with principal direction $du_0 : dv_0$ if and only if $\kappa_0' du_0, dv_0$ satisfy

$$\begin{aligned} (L - \kappa_0 E) du_0 + (M - \kappa_0 F) dv_0 &= 0 \\ (M - \kappa_0 F) du_0 + (N - \kappa_0 G) dv_0 &= 0 \end{aligned} \quad (a)$$

Suppose κ_0 is a principal curvature with associated principal direction $du_0 : dv_0$. Recall that the principal curvatures are the maximum or minimum values of the normal curvature κ_n . Thus if

$$\kappa_n = \frac{II}{I} = \frac{L du^2 + 2M du dv + N dv^2}{E du^2 + 2F du dv + G dv^2}$$

takes on a maximum or minimum κ_0 at (du_0, dv_0) , then, from calculus, the partial derivatives

$$\frac{\partial \kappa_n}{\partial du} \Big|_{(du_0, dv_0)} = 0 \quad \text{and} \quad \frac{\partial \kappa_n}{\partial dv} \Big|_{(du_0, dv_0)} = 0$$

or, differentiating,

$$I \frac{II_{du} - II I_{du}}{I^2} \Big|_{(du_0, dv_0)} = 0 \quad \text{and} \quad \frac{I II_{dv} - II I_{dv}}{I^2} \Big|_{(du_0, dv_0)} = 0$$

Multiplying by I gives

$$II_{du} - \frac{II}{I} I_{du} \Big|_{(du_0, dv_0)} = 0 \quad \text{and} \quad II_{dv} - \frac{II}{I} I_{dv} \Big|_{(du_0, dv_0)} = 0$$

Hence

$$\text{But } (II/I)|_{(du_0, dv_0)} = \kappa_n|_{(du_0, dv_0)} = \kappa_0. \quad \text{and} \quad (II_{dv} - \kappa_0 I_{dv})|_{(du_0, dv_0)} = 0$$

$$(II_{du} - \kappa_0 I_{du})|_{(du_0, dv_0)} = 0 \quad \text{and} \quad I_{du} = 2E du + 2F dv, \text{ etc.,}$$

$$\text{Since } II_{du} = 2L du + 2M dv \text{ and } I_{du} = 2E du + 2F dv, \text{ etc.,}$$

$$(L du_0 + M dv_0) - \kappa_0(E du_0 + F dv_0) = 0$$

$$(M du_0 + N dv_0) - \kappa_0(F du_0 + G dv_0) = 0$$

$$(M du_0 + N dv_0) - \kappa_0(F du_0 + G dv_0) = 0$$

which gives the required result. Now conversely, suppose that κ_0 , du_0 , dv_0 , $du_0^2 + dv_0^2 \neq 0$, satisfy equation (a) above. Then κ_0 together with the principal curvatures must satisfy

$$\det \begin{pmatrix} L - \kappa E & M - \kappa F \\ M - \kappa F & N - \kappa G \end{pmatrix} = 0$$

or, expanding,

$$(EG - F^2)\kappa^2 - (EN + GL - 2FM)\kappa + (LN - M^2) = 0$$

Now suppose P is an umbilical point with curvature κ . Since κ is taken in every direction, the coefficients of (a) must all be zero, i.e. $\kappa = L/E = M/F = N/G$. But then it follows from Problem 9.14 that equation (b) has a single root with multiplicity two and therefore $\kappa = \kappa_0$. Thus κ_0 is the principal curvature, and every direction including $du_0:dv_0$ is a principal direction. If P is a nonumbilical point, κ_0 must be one of the two distinct roots of (b), i.e. one of the two principal curvatures at a nonumbilical point with principal direction $du_0:dv_0$, which proves the theorem.

9.17. Show that the curvature κ at a point P on the curve C of intersection of two surfaces satisfies

$$\kappa^2 \sin^2 \alpha = \kappa_1^2 + \kappa_2^2 - 2\kappa_1\kappa_2 \cos \alpha$$

where κ_1 and κ_2 are the normal curvatures of the surfaces in the direction of C at P , and α is the angle between the normals to the surfaces at P .

From equation (9.15), page 179,

$$\kappa_1 N_2 = \kappa(n \cdot N_1)N_2 \quad \text{and} \quad \kappa_2 N_1 = \kappa(n \cdot N_2)N_1$$

Subtracting and using the vector identity of Theorem 1.8, page 10,

$$\kappa_1 N_2 - \kappa_2 N_1 = \kappa[(n \cdot N_1)N_2 - (n \cdot N_2)N_1] = \kappa(N_1 \times N_2) \times n$$

It follows further, using the identity $[F_1]$, page 10, that

$$\begin{aligned} (\kappa_1 N_2 - \kappa_2 N_1) \cdot (\kappa_1 N_2 - \kappa_2 N_1) &= \kappa^2[(N_1 \times N_2) \times n] \cdot [(N_1 \times N_2) \times n] \\ &= \kappa^2[(N_1 \times N_2) \cdot (N_1 \times N_2) - (N_1 \times N_2) \cdot n]^2 \\ &= \kappa^2(N_1 \times N_2) \cdot (N_1 \times N_2) \end{aligned}$$

where we used the fact that $N_1 \times N_2$ is a vector parallel to the tangent to the curve and hence $(N_1 \times N_2) \cdot n = 0$. Expanding,

$$\kappa_1^2(N_2 \cdot N_2) - 2\kappa_1\kappa_2(N_1 \cdot N_2) + \kappa_2^2(N_1 \cdot N_1) = \kappa^2|N_1 \times N_2|^2$$

or

$$\kappa_1^2 - 2\kappa_1\kappa_2 \cos \alpha + \kappa_2^2 = \kappa^2 \sin^2 \alpha$$

9.18. Prove that at each point on a patch,

$$N_u \times N_v = K(x_u \times x_v)$$

where K is the Gaussian curvature at the point.

Since N is a function of unit length, N_u and N_v are orthogonal to N and hence parallel to the tangent plane. It follows that we can write $N_u = ax_u + bx_v$ and $N_v = cx_u + dx_v$, where a, b, c, d are to be determined. Note that

$$N_u \times N_v = (ax_u + bx_v) \times (cx_u + dx_v) = (ad - bc)(x_u \times x_v)$$

Thus it remains to show that

$$ad - bc = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = K$$

From the above and (9.10), page 175, we have

$$x_u \cdot N_u = ax_u \cdot x_u + bx_u \cdot x_v = aE + bF = -L$$

Similarly,

$$\mathbf{x}_v \cdot \mathbf{N}_u = aF + bG = -M$$

$$\mathbf{x}_u \cdot \mathbf{N}_v = cE + dF = -M$$

$$\mathbf{x}_v \cdot \mathbf{N}_v = cF + dG = -N$$

These equations can be written as the matrix product

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} -L & -M \\ -M & -N \end{pmatrix}$$

$$\text{Hence} \quad \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \det \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \det \begin{pmatrix} -L & -M \\ -M & -N \end{pmatrix}$$

or

$$ad - cb = \frac{LN - M^2}{EG - F^2} = K$$

which completes the proof.

LINES OF CURVATURE

919. Determine the principal directions to $\mathbf{x} = ue_1 + ve_2 + (u^2 + v^2)e_3$ at $u = 1$, $v = 1$ and verify Rodrigues' formula in each direction.

From Example 9.11, page 185, $E = 1 + 4u^2$, $F = 4uv$, $G = 1 + 4v^2$, $L = 2(4u^2 + 4v^2 + 1)^{-1/2}$, $M = 0$, $N = 2(4u^2 + 4v^2 + 1)^{-1/2}$. At $u = 1$, $v = 1$, we have $E = 5$, $F = 4$, $G = 5$, $L = 2/3$, $M = 0$, $N = 2/3$. From equation (9.25), page 185, the principal directions at $u = 1$, $v = 1$ are the solutions of

$$-\frac{8}{3}du^2 + \frac{8}{3}dv^2 = 0 \quad \text{or} \quad (du + dv)(du - dv) = 0$$

Hence $du_1:dv_1 = 1:-1$ and $du_2:dv_2 = 1:1$. We have further,

$$\mathbf{x}_u = \mathbf{e}_1 + 2ue_3, \quad \mathbf{x}_v = \mathbf{e}_2 + 2ve_3$$

$$\mathbf{N} = (4u^2 + 4v^2 + 1)^{-1/2}(-2ue_1 - 2ve_2 + e_3)$$

$$\mathbf{N}_u = (4u^2 + 4v^2 + 1)^{-3/2}[-(8v^2 + 2)e_1 + 8uve_2 - 4ue_3]$$

$$\mathbf{N}_v = (4u^2 + 4v^2 + 1)^{-3/2}[8uve_1 - (8v^2 + 2)e_2 - 4ve_3]$$

At $u = 1$, $v = 1$,

$$\mathbf{x}_u = \mathbf{e}_1 + 2e_3, \quad \mathbf{x}_v = \mathbf{e}_2 + 2e_3, \quad \mathbf{N}_u = \frac{1}{27}(-10e_1 + 8e_2 - 4e_3), \quad \mathbf{N}_v = \frac{1}{27}(8e_1 - 10e_2 - 4e_3)$$

$$d\mathbf{N}_1 = \mathbf{N}_u du_1 + \mathbf{N}_v dv_1 = \frac{1}{27}(-18e_1 + 18e_2) = -\frac{18}{27}(e_1 - e_2), \quad d\mathbf{x}_1 = \mathbf{x}_u du_1 + \mathbf{x}_v dv_1 = \mathbf{e}_1 - \mathbf{e}_2$$

Thus $d\mathbf{N}_1 = -\frac{18}{27}d\mathbf{x}_1$. Also,

$$d\mathbf{N}_2 = \mathbf{N}_u du_2 + \mathbf{N}_v dv_2 = \frac{1}{27}[-2e_1 - 2e_2 - 8e_3] = -\frac{2}{27}[\mathbf{e}_1 + \mathbf{e}_2 + 4e_3]$$

$$d\mathbf{x}_2 = \mathbf{x}_u du_2 + \mathbf{x}_v dv_2 = \mathbf{e}_1 + \mathbf{e}_2 + 4e_3$$

Hence $d\mathbf{N}_2 = -\frac{2}{27}d\mathbf{x}_2$, verifying Rodrigues' formula.

920. Prove that the solutions to $A du^2 + 2B du dv + C dv^2 = 0$ form orthogonal families of curves on a patch if and only if $EC - 2FB + GA = 0$.

Suppose

$$A du^2 + 2B du dv + C dv^2 = (A' du + B' dv)(C' du + D' dv)$$

Then one of the family of curves is the solution to $A' du + B' dv = 0$ and the other is the solution to, say, $C' du + D' dv = 0$. Along the first, $du:dv = B':-A'$; and along the second, $du:dv = D':-C'$. But from Theorem 9.1, page 173, the families are orthogonal if and only if

$$D':-C' = EC - 2FB + GA = 0$$

9.21. Prove Theorem 9.11. That is, prove that if P is a point on a surface of class ∞^2 , then there exists a coordinate patch containing P such that the directions of the parameter lines are principal directions.

It is sufficient to consider the case where P is a nonumbilical point. For otherwise every direction is a principal direction and any patch containing P will suffice. Now suppose $\mathbf{x} = \mathbf{x}(u, v)$ is an arbitrary patch containing the nonumbilical point P and suppose $du_1:dv_1$ and $du_2:dv_2$ are the principal directions at P . Consider the linear parameter transformation $u = du_1\theta + du_2\phi$, $v = dv_1\theta + dv_2\phi$. Note that $\frac{\partial(u, v)}{\partial(\theta, \phi)} = \det \begin{pmatrix} du_1 & du_2 \\ dv_1 & dv_2 \end{pmatrix} \neq 0$, since the directions $du_1:dv_1$ and $du_2:dv_2$ are distinct. Thus the parameter transformation is an allowable transformation of class C^∞ . It follows that $\mathbf{x} = \mathbf{x}^*(\theta, \phi) = \mathbf{x}(u(\theta, \phi), v(\theta, \phi))$ is a patch of class C^2 containing P and that

$$\mathbf{x}_\theta = \mathbf{x}_u(\partial u/\partial \theta) + \mathbf{x}_v(\partial v/\partial \theta) = \mathbf{x}_u du_1 + \mathbf{x}_v dv_1$$

$$\mathbf{x}_\phi = \mathbf{x}_u(\partial u/\partial \phi) + \mathbf{x}_v(\partial v/\partial \phi) = \mathbf{x}_u du_2 + \mathbf{x}_v dv_2$$

Namely the θ and ϕ parameter curves at P are in the direction of the principal directions.

9.22. *Euler's Theorem.* Prove that the normal curvature at a point on a surface of class ∞^2 in the direction of a tangent line L is given by

$$\kappa_n = \kappa_1 \cos^2 \alpha + \kappa_2 \sin^2 \alpha$$

where κ_1 and κ_2 are the principal curvatures at P and α is the angle between L and a tangent line in the principal direction corresponding to κ_1 .

The theorem is clearly true if P is an umbilical point where $\kappa_1 = \kappa_2 = \kappa_n$. Otherwise let $\mathbf{x} = \mathbf{x}(u, v)$ be a patch containing P such that the u - and v -parameter curves are in the direction of the principal directions. Then from Theorem 9.12, page 186, $F = M = 0$ and the normal curvature in any direction $du:dv$ is $\kappa_n = \frac{L du^2 + N dv^2}{E du^2 + G dv^2}$. From Theorem 9.13 it follows that the principal curvatures are $\kappa_1 = L/E$ and $\kappa_2 = N/G$ respectively. Substituting,

$$\kappa_n = \kappa_1 \frac{E du^2}{E du^2 + G dv^2} + \kappa_2 \frac{G dv^2}{E du^2 + G dv^2}$$

Now if α and β are the angles between an arbitrary tangent line L with direction numbers $du:dv$ and the tangent lines in the principal directions 1:0 and 0:1 respectively, we have from equation (9.6), page 173,

$$\cos \alpha = \frac{E du}{\sqrt{E du^2 + G dv^2} \sqrt{E}} \quad \text{and} \quad \cos \beta = \frac{G dv}{\sqrt{E du^2 + G dv^2} \sqrt{G}}$$

Squaring and substituting,

$$\kappa_n = \kappa_1 \cos^2 \alpha + \kappa_2 \cos^2 \beta$$

But the principal directions are perpendicular, i.e. $\beta = \pi/2 - \alpha$. Hence $\kappa_n = \kappa_1 \cos^2 \alpha + \kappa_2 \sin^2 \alpha$.

9.23. Complete the proof of Theorem 9.14. That is, prove that $du:dv$ is a principle direction at a point on a patch if for some scalar κ

$dN = -\kappa d\mathbf{x}$

in the direction $du:dv$.

From $dN = -\kappa d\mathbf{x}$ we have $(dN + \kappa d\mathbf{x}) \cdot \mathbf{x}_u = 0$ and $(dN + \kappa d\mathbf{x}) \cdot \mathbf{x}_v = 0$ or

$$[(N_u du + N_v dv) + \kappa(\mathbf{x}_u du + \mathbf{x}_v dv)] \cdot \mathbf{x}_u = 0$$

$$[(N_u du + N_v dv) + \kappa(\mathbf{x}_u du + \mathbf{x}_v dv)] \cdot \mathbf{x}_v = 0$$

or

$$(-N_u \cdot \mathbf{x}_u - \kappa \mathbf{x}_u \cdot \mathbf{x}_u) du + (-N_v \cdot \mathbf{x}_u - \kappa \mathbf{x}_v \cdot \mathbf{x}_u) dv = 0$$

$$(-N_u \cdot \mathbf{x}_v - \kappa \mathbf{x}_u \cdot \mathbf{x}_v) du + (-N_v \cdot \mathbf{x}_v - \kappa \mathbf{x}_v \cdot \mathbf{x}_v) dv = 0$$

or

$$(L - \kappa E) du + (M - \kappa F) dv = 0$$

$$(M - \kappa F) du + (N - \kappa G) dv = 0$$

From Theorem 9.5, page 183, it follows that κ is a principal curvature and $du:dv$ is the corresponding principal direction.

9.24. Prove Theorem 9.9. That is, prove that $du:dv$ is a principal direction at a point on a patch if and only if du and dv satisfy

$$(EM - LF) du^2 + (EN - LG) du dv + (FN - MG) dv^2 = 0$$

From Theorem 9.5, $du:dv$ is a principal direction if and only if for some κ ,

$$(L - \kappa E) du + (M - \kappa F) dv = 0$$

$$(M - \kappa F) du + (N - \kappa G) dv = 0$$

or

$$(L du + M dv) - \kappa(E du + F dv) = 0$$

$$(M du + N dv) - \kappa(F du + G dv) = 0$$

But the above can have a nontrivial solution $(1, -\kappa)$ if and only if

$$\det \begin{pmatrix} L du + M dv & E du + F dv \\ M du + N dv & F du + G dv \end{pmatrix} = 0$$

or, expanding,

$$(EM - LF) du^2 + (EN - LG) du dv + (FN - MG) dv^2 = 0$$

9.25. If two surfaces intersect at a constant angle and if the curve of intersection is a line of curvature on one of the surfaces, prove that it is a line of curvature on the other surface.

Since the surfaces, say M_1 and M_2 , intersect at a constant angle, along the curve of intersection $N_1 \cdot N_2 = \text{constant}$. Hence

$$0 = \frac{d}{dt}(N_1 \cdot N_2) = \left(\frac{d}{dt} N_1 \right) \cdot N_2 + N_1 \cdot \frac{d}{dt} N_2$$

Assuming the curve of intersection is a line of curvature along M_1 , then (Rodrigues' formula)

$$\frac{dN_1}{dt} = -\kappa_1 \frac{dx}{dt} \quad \text{Hence} \quad -\kappa_1 \frac{dx}{dt} \cdot N_2 + N_1 \cdot \frac{dN_2}{dt} = 0$$

But dx/dt is orthogonal to N_2 , i.e. $(dx/dt) \cdot N_2 = 0$. Thus $N_1 \cdot (dN_2/dt) = 0$. Hence dN_2/dt is orthogonal to N_1 . But dN_2/dt is also orthogonal to N_2 , since N_2 is of unit length. It follows that dN_2/dt is parallel to dx/dt . Namely, there exists κ_2 such that $dN_2/dt = -\kappa_2(dx/dt)$. Hence the curve of intersection is also a line of curvature along M_2 .

9.26. *Third Fundamental Form.* The third fundamental form is defined by $\text{III} = dN \cdot dN$. Prove that $\text{III} - 2H\text{II} + K\text{I} = 0$ where H and K are the mean and Gaussian curvatures respectively.

It is easily verified that III is invariant, in the same sense as I. Note that II and H change sign with a change in orientation. Thus it is sufficient to consider a fixed point P and a coordinate patch containing P such that the directions of the u - and v -parameter curves at P are principal directions. From Rodrigues' formula,

$$N_u = -\kappa_1 x_u \quad \text{and} \quad N_v = -\kappa_2 x_v$$

It follows that for arbitrary (du, dv) ,

$$\begin{aligned} \text{where } \kappa_1 \text{ and } \kappa_2 \text{ are the principal curvatures.} \\ dN = N_u du + N_v dv &= -\kappa_1 x_u du - \kappa_2 x_v dv \\ &= -\kappa_1 x_u du - \kappa_1 x_v dv + \kappa_1 x_v dv - \kappa_2 x_v dv = -\kappa_1 dx + (\kappa_1 - \kappa_2) x_v dv \\ &= -\kappa_1 x_u du - \kappa_1 x_v dv + \kappa_1 dx = (\kappa_1 - \kappa_2) x_v dv \end{aligned}$$

or

$$\text{Also, } dN = -\kappa_1 x_u du + \kappa_2 x_u du - \kappa_2 x_u du - \kappa_2 x_v dv = -\kappa_2 dx + (\kappa_2 - \kappa_1) x_u dv$$

$$\text{or } dN + \kappa_2 dx = (\kappa_2 - \kappa_1) x_u dv$$

$$\text{Thus } (dN + \kappa_1 dx) \cdot (dN + \kappa_2 dx) = (\kappa_1 - \kappa_2)(\kappa_2 - \kappa_1) du dv x_u \cdot x_v$$

Since the lines of curvature are orthogonal at P , we have $x_u \cdot x_v = 0$. Then

$$(dN + \kappa_1 dx) \cdot (dN + \kappa_2 dx) = 0$$

$$\text{or } dN \cdot dN + (\kappa_1 + \kappa_2) dN \cdot dx + \kappa_1 \kappa_2 dx \cdot dx = 0$$

$$\text{Hence III} - 2H\text{II} + K\text{I} = 0.$$

9.27. Show that the principal directions on a surface given by $f(x_1, x_2, x_3) = C$ are solutions to

$$\det \begin{pmatrix} dx_1 & f_{x_1} & df_{x_1} \\ dx_2 & f_{x_2} & df_{x_2} \\ dx_3 & f_{x_3} & df_{x_3} \end{pmatrix} = 0, \quad f_{x_1} dx_1 + f_{x_2} dx_2 + f_{x_3} dx_3 = 0$$

If $dx = dx_1 e_1 + dx_2 e_2 + dx_3 e_3$ is tangent, then $f_{x_1} dx_1 + f_{x_2} dx_2 + f_{x_3} dx_3 = 0$. Hence $G = f_{x_1} e_1 + f_{x_2} e_2 + f_{x_3} e_3$ is normal, and $N = G/|G|$. Thus

$$dN = \frac{dG}{|G|} - \frac{(G \cdot dG)G}{|G|^3}$$

where $dG = df_{x_1} e_1 + df_{x_2} e_2 + df_{x_3} e_3$. Now suppose $dx = dx_1 e_1 + dx_2 e_2 + dx_3 e_3$ is a principal direction. Then from Rodrigues' formula, dx is parallel to dN . Hence the three vectors dx , N and dN are dependent, or

$$0 = [dx \ N \ dN] = \left[dx \ \frac{G}{|G|} \left(\frac{dG}{|G|} - \frac{(G \cdot dG)G}{|G|^3} \right) \right] = \frac{1}{|G|^2} [dx \ G \ dG] - \frac{G \cdot dG}{|G|^4} [dx \ G \ G]$$

But $[dx \ G \ G] = 0$; hence $[dx \ G \ dG] = 0$. The required result follows.

ASYMPTOTIC LINES — CONJUGATE FAMILIES OF CURVES

9.28. Prove that the parameter curves on the surface $x = x_1(u) + x_2(v)$ where x_1 and x_2 are arbitrary, are conjugate families of curves.

$x_u = x'_1(u)$, $x_v = x'_2(v)$, $x_{uv} \equiv 0$. Hence $M = x_{uv} \cdot N \equiv 0$. Thus from Theorem 9.20, page 183, the parameter curves are conjugate families.

9.29. Prove Theorem 9.19; that is, prove that at an elliptic or hyperbolic point every direction has a unique conjugate direction.

From equation (9.29), page 188, $\delta u : \delta v$ is conjugate to $du : dv$ if and only if

$$(L du + M dv) \delta u + (M du + N dv) \delta v = 0$$

The above equation has a unique solution $\delta u : \delta v$, $\delta u^2 + \delta v^2 \neq 0$, if and only if the coefficients do not both vanish. Thus, given $du : dv$, there exists a unique $\delta u : \delta v$ if and only if

$$(L du + M dv)^2 + (M du + N dv)^2 \neq 0$$

or

$$(L^2 + M^2) du^2 + 2(LM + MN) du dv + (M^2 + N^2) dv^2 \neq 0$$

But this equation is different from zero for all $du : dv$, $du^2 + dv^2 \neq 0$, if and only if its discriminant

$$(L^2 + M^2)(M^2 + N^2) - (LM + MN)^2 > 0$$

or, expanding, $(LN - M^2)^2 > 0$ or $LN - M^2 \neq 0$. Thus every direction $du : dv$ has a unique conjugate direction, $\delta u : \delta v$ if and only if $LN - M^2 \neq 0$, which proves the theorem and in fact its converse.

930.

Prove Theorem 9.18; that is, prove that the torsion along an asymptotic line which is not a straight line satisfies $\tau^2 = -K$.

From Theorem 9.17, page 188, the osculating plane at each point on the asymptotic line is tangent to the surface. Hence along the curve the binormal $\mathbf{b} = \pm \mathbf{N}$. It follows that $KI = 0$. But along an asymptotic line $II = 0$, $III = (dN/ds) \cdot (dN/ds) = \tau^2(\mathbf{n} \cdot \mathbf{n}) = \tau^2$. From Problem 9.26, $III = 2HI +$ Hence $\tau^2 + K = 0$ or $\tau^2 = -K$.

931.

Show that the asymptotic directions on a surface given by $f(x_1, x_2, x_3) = C$ are solutions of

$$dx_1 df_{x_1} + dx_2 df_{x_2} + dx_3 df_{x_3} = 0, \quad f_{x_1} dx_1 + f_{x_2} dx_2 + f_{x_3} dx_3 = 0$$

From Problem 9.27, $\mathbf{G} = f_{x_1} \mathbf{e}_1 + f_{x_2} \mathbf{e}_2 + f_{x_3} \mathbf{e}_3$ is normal to the surface. $\mathbf{N} = \frac{\mathbf{G}}{|\mathbf{G}|}$ and $dN = \frac{d\mathbf{G}}{|\mathbf{G}|} - \frac{(\mathbf{G} \cdot d\mathbf{G})\mathbf{G}}{|\mathbf{G}|^3}$. Hence if $d\mathbf{x} = dx_1 \mathbf{e}_1 + dx_2 \mathbf{e}_2 + dx_3 \mathbf{e}_3$ is in an asymptotic direction,

$$0 = II = -d\mathbf{x} \cdot d\mathbf{N} = -\frac{d\mathbf{x} \cdot d\mathbf{G}}{|\mathbf{G}|} - \frac{(\mathbf{G} \cdot d\mathbf{G})(d\mathbf{x} \cdot \mathbf{G})}{|\mathbf{G}|^3}$$

But $d\mathbf{x} \cdot \mathbf{G} = 0$. Thus $d\mathbf{x} \cdot d\mathbf{G} = 0$, or $dx_1 df_{x_1} + dx_2 df_{x_2} + dx_3 df_{x_3} = 0$.

932.

Use the above to find the asymptotic curves on the surface $f = x_3 - x_1 \sin x_2 = 0$, $-\pi/2 < x_2 < \pi/2$, $x_1 > 0$.

$f_{x_1} = -\sin x_2$, $f_{x_2} = -x_1 \cos x_2$, $f_{x_3} = 1$, $df_{x_1} = -\cos x_2 dx_2$, $df_{x_2} = -\cos x_2 dx_1 + x_1 \sin x_2 dx_2$, $df_{x_3} = 0$. The equation

$$df_{x_1} dx_1 + df_{x_2} dx_2 + df_{x_3} dx_3 = -2 \cos x_2 dx_1 dx_2 + x_1 \sin x_2 dx_2^2 = 0$$

has factors $dx_2 = 0$ and $-2 \cos x_2 dx_1 + x_1 \sin x_2 dx_2 = 0$ which have solutions $x_2 = C$ and $x_1 = K \sec^{1/2} x_2$. Substituting the first into $x_3 - x_1 \sin x_2 = 0$ gives the family of straight lines $x_1 = t$, $x_2 = C$, $x_3 = t \sin C$, $t > 0$. The second gives the family of curves $x_1 = K \sec^{1/2} u$, $x_2 = u$, $x_3 = K \sec^{1/2} u \sin u$, $-\pi/2 < u < \pi/2$.

Supplementary Problems

933. Show that the parameter curves on a Monge patch $\mathbf{x} = u\mathbf{e}_1 + v\mathbf{e}_2 + f(u, v)\mathbf{e}_3$ are an orthogonal family of curves if and only if $f_u f_v = 0$.

934.

Show that the first fundamental form on the tangent surface $\mathbf{x} = \mathbf{y}(s) + u\mathbf{t}(s)$ to the curve $\mathbf{y} = \mathbf{y}(s)$ is $I = (1 + u^2 \kappa^2) ds^2 + 2ds du + du^2$.

935.

Show that the first fundamental form of the surface $\mathbf{x} = \mathbf{y}(s) + u\mathbf{b}(s)$ generated by the binormal $\mathbf{b}(s)$ of a curve $\mathbf{y} = \mathbf{y}(s)$ is $I = (1 + u^2 \tau^2) ds^2 + du^2$.

936.

Find the length of the arc $\theta = \int_{\pi/4}^t \frac{1}{\sin \tau} d\tau$, $\phi = t$, $\pi/4 \leq t \leq \pi/2$, on the sphere $\mathbf{x} = (\sin \phi \cos \theta)\mathbf{e}_1 + (\sin \phi \sin \theta)\mathbf{e}_2 + (\cos \phi)\mathbf{e}_3$.
Ans. $\sqrt{2}\pi/4$

937.

Show that the surface area on a Monge patch $\mathbf{x} = u\mathbf{e}_1 + v\mathbf{e}_2 + f(u, v)\mathbf{e}_3$ is given by the integral

$$A = \iint_W \sqrt{1 + f_u^2 + f_v^2} du dv.$$

938.

Prove that on the intersection of two coordinate patches $\mathbf{x} = \mathbf{x}(u, v)$ and $\mathbf{x} = \mathbf{x}^*(\theta, \phi)$ on a surface,

$$EG - F^2 = (E^*G^* - F^{*2}) \left[\frac{\partial(\theta, \phi)}{\partial(u, v)} \right]^2.$$

- 9.39. Show that the curves on the surface $\mathbf{x} = (u \cos \theta)\mathbf{e}_1 + (u \sin \theta)\mathbf{e}_2 + (a\theta + b)\mathbf{e}_3$ satisfying $(u^2 + a^2)d\theta^2 - du^2 = 0$ are orthogonal families of curves.
- 9.40. Show that the θ -parameter curves on the surface $\mathbf{x} = (r \cos \theta)\mathbf{e}_1 + (r \sin \theta)\mathbf{e}_2 + f(\theta)\mathbf{e}_3$ are parallel.
- 9.41. Show that the second fundamental form on a Monge patch $\mathbf{x} = u\mathbf{e}_1 + v\mathbf{e}_2 + f(u, v)\mathbf{e}_3$ is
- $$\text{II} = (f_u^2 + f_v^2 + 1)^{-1/2} [f_{uu} du^2 + 2f_{uv} du dv + f_{vv} dv^2]$$
- 9.42. Show that all points on the general cylinder $\mathbf{x} = y(s) + u\mathbf{g}$, $\mathbf{g} = \text{constant}$, are parabolic or planar.
- 9.43. If $\mathbf{x} = \mathbf{x}(u, v)$, $\mathbf{x} = \mathbf{x}^*(\theta, \phi)$ are patches on a surface such that on the intersection $\partial(\theta, \phi)/\partial(u, v) > 0$, show that the second fundamental coefficients transform as follows:
- $$\begin{aligned} L &= L^*\theta_u^2 + 2M^*\theta_u\phi_u + N^*\phi_u^2 \\ M &= L^*\theta_u\phi_v + M^*(\theta_u\phi_v + \phi_u\theta_v) + N^*\phi_u\phi_v \\ N &= L^*\phi_v^2 + 2M^*\theta_v\phi_v + N^*\phi_v^2 \end{aligned}$$
- 9.44. Show that the Gauss and mean curvatures on $\mathbf{x} = (u + v)\mathbf{e}_1 + (u - v)\mathbf{e}_2 + uv\mathbf{e}_3$ at $u = 1$, $v = 1$ are $K = 1/16$ and $H = 1/8\sqrt{2}$.
- 9.45. Show that the mean curvature is zero at every point on the surface of revolution
- $$\mathbf{x} = (\cosh u \cos \theta)\mathbf{e}_1 + (\cosh u \sin \theta)\mathbf{e}_2 + u\mathbf{e}_3$$
- 9.46. Show that the principal curvatures of the surface $x_1 \sin x_3 - x_2 \cos x_3 = 0$ are $\pm 1/(x_1^2 + x_2^2 + 1)$.
- 9.47. Show that the lines of curvature on the surface $\mathbf{x} = (u \cos \theta)\mathbf{e}_1 + (u \sin \theta)\mathbf{e}_2 + \theta\mathbf{e}_3$ are the images of $\log(\theta + \sqrt{u^2 + 1}) - v = C$ and $\log(\theta + \sqrt{u^2 + 1}) + v = K$.
- 9.48. Find the principal curvatures and principal directions on
- $$\mathbf{x} = u\mathbf{e}_1 + v\mathbf{e}_2 + (4u^2 + v^2)\mathbf{e}_3$$
- at $u = 0$, $v = 0$ using Dupin's indicatrix.
- 9.49. Prove that the family of curves on a patch which are orthogonal to the family of curves given by $A du + B dv = 0$ are given by the solutions to $(EB - FA) du + (FB - GA) dv = 0$.
- 9.50. The parameter curves on the surface
- $$\mathbf{x} = e^{(u-v)/2} \left(\cos \frac{u+v}{2} \right) \mathbf{e}_1 + e^{(u-v)/2} \left(\sin \frac{u+v}{2} \right) \mathbf{e}_2 + \left(\frac{u-v}{2} \right) \mathbf{e}_3$$
- are asymptotic lines. Verify that along the u -parameter curve $v = 0$, the torsion satisfies $\tau^2 = -K$.
- 9.51. Show that the asymptotic lines on the surface $x_3 - x_1^4 + x_2^4 = 0$ are the intersections of the surface and the families of cylinders $x_1^2 + x_2^2 = C$ and $x_1^2 - x_2^2 = K$.
- 9.52. Prove that the directions of curvature bisect the asymptotic directions.
- 9.53. Show that the mean curvature is zero on a surface whose asymptotic lines are orthogonal families of curves.
- 9.54. If a sphere or a plane intersects a surface at a constant angle, prove that curve of intersection is a line of curvature.
- 9.55. Prove that the sum of the normal curvatures at a point on a surface in any pair of orthogonal directions is constant.
- 9.56. If a surface has a one-parameter family of plane asymptotic curves other than straight lines, prove that the surface is a plane.
- 9.57. Suppose R is a region on a patch on a surface. The endpoints of the unit normals in R form a set R' on the unit sphere called the spherical image of R . Show that the ratio of the area of R' to the area of R tends to $|K|$ at a point P when R shrinks down to the point P . *Hint.* Problem 9.18, page 194.
- 9.58. If $K \neq 0$ at a point P on a surface, show that there is a neighborhood of P in which the points can be put into a 1-1 correspondence with the spherical image of the neighborhood (see Problem 9.57).

Chapter 10

Theory of Surfaces Tensor Analysis

GAUSS-WEINGARTEN EQUATIONS

The Gauss-Weingarten equations for surfaces are analogous to the Frenet equations for curves. We recall that the Frenet equations express the vectors $\mathbf{t}$, $\mathbf{n}$ and $\mathbf{b}$ as linear combinations of $\mathbf{t}$, $\mathbf{n}$ and $\mathbf{b}$ with coefficients depending upon κ and τ . Similarly the Gauss-Weingarten equations express the derivatives of the vectors $\mathbf{x}_u$, $\mathbf{x}_v$ and $\mathbf{N}$ as linear combinations of these vectors with coefficients which are functions of the first and second fundamental coefficients.

We assume that $\mathbf{x} = \mathbf{x}(u, v)$ is a patch on a surface of class ≥ 2 . Then $\mathbf{x}_u$, $\mathbf{x}_v$ and $\mathbf{N}$ are functions of class C^1 and have continuous derivatives $\mathbf{x}_{uu}$, $\mathbf{x}_{uv}$, $\mathbf{x}_{vv}$, $\mathbf{N}_u$ and $\mathbf{N}_v$. Since the vectors $\mathbf{x}_u$, $\mathbf{x}_v$ and $\mathbf{N}$ are linearly independent, we can write

$$\begin{aligned}\mathbf{x}_{uu} &= \Gamma_{11}^1 \mathbf{x}_u + \Gamma_{11}^2 \mathbf{x}_v + \alpha_{11} \mathbf{N} \\ \mathbf{x}_{uv} &= \Gamma_{12}^1 \mathbf{x}_u + \Gamma_{12}^2 \mathbf{x}_v + \alpha_{12} \mathbf{N} \\ \mathbf{x}_{vv} &= \Gamma_{22}^1 \mathbf{x}_u + \Gamma_{22}^2 \mathbf{x}_v + \alpha_{22} \mathbf{N} \\ \mathbf{N}_u &= \beta_1^1 \mathbf{x}_u + \beta_1^2 \mathbf{x}_v + \gamma_1 \mathbf{N} \\ \mathbf{N}_v &= \beta_2^1 \mathbf{x}_u + \beta_2^2 \mathbf{x}_v + \gamma_2 \mathbf{N}\end{aligned}\quad (10.1)$$

where the coefficients Γ_{ij}^k , α_{ij} , β_i^j , γ_i are to be determined.

Since $\mathbf{N}$ is of unit length, $\mathbf{N}_u$ and $\mathbf{N}_v$ are orthogonal to $\mathbf{N}$. Hence

$$\begin{aligned}0 &= \mathbf{N}_u \cdot \mathbf{N} = \beta_1^1 \mathbf{x}_u \cdot \mathbf{N} + \beta_1^2 \mathbf{x}_v \cdot \mathbf{N} + \gamma_1 \mathbf{N} \cdot \mathbf{N} \\ 0 &= \mathbf{N}_v \cdot \mathbf{N} = \beta_2^1 \mathbf{x}_u \cdot \mathbf{N} + \beta_2^2 \mathbf{x}_v \cdot \mathbf{N} + \gamma_2 \mathbf{N} \cdot \mathbf{N}\end{aligned}$$

But $\mathbf{x}_u \cdot \mathbf{N} = \mathbf{x}_v \cdot \mathbf{N} = 0$ and $\mathbf{N} \cdot \mathbf{N} = 1$. Thus $\gamma_1 = \gamma_2 = 0$. It follows further that

$$\begin{aligned}-L &= \mathbf{x}_u \cdot \mathbf{N}_u = \beta_1^1 \mathbf{x}_u \cdot \mathbf{x}_u + \beta_1^2 \mathbf{x}_u \cdot \mathbf{x}_v = \beta_1^1 E + \beta_1^2 F \\ -M &= \mathbf{x}_v \cdot \mathbf{N}_u = \beta_1^1 \mathbf{x}_v \cdot \mathbf{x}_u + \beta_1^2 \mathbf{x}_v \cdot \mathbf{x}_v = \beta_1^1 F + \beta_1^2 G \\ -M &= \mathbf{x}_u \cdot \mathbf{N}_v = \beta_2^1 \mathbf{x}_u \cdot \mathbf{x}_u + \beta_2^2 \mathbf{x}_u \cdot \mathbf{x}_v = \beta_2^1 E + \beta_2^2 F \\ -N &= \mathbf{x}_v \cdot \mathbf{N}_v = \beta_2^1 \mathbf{x}_v \cdot \mathbf{x}_u + \beta_2^2 \mathbf{x}_v \cdot \mathbf{x}_v = \beta_2^1 F + \beta_2^2 G\end{aligned}$$

Solving the first two equations for β_1^1 and β_1^2 and the second two for β_2^1 and β_2^2 gives

$$\beta_1^1 = \frac{MF - LG}{EG - F^2}, \quad \beta_1^2 = \frac{LF - ME}{EG - F^2}, \quad \beta_2^1 = \frac{NF - MG}{EG - F^2}, \quad \beta_2^2 = \frac{MF - NE}{EG - F^2} \quad (10.2)$$

Continuing, we obtain

$$\begin{aligned}L &= \mathbf{x}_{uu} \cdot \mathbf{N} = \Gamma_{11}^1 \mathbf{x}_u \cdot \mathbf{N} + \Gamma_{11}^2 \mathbf{x}_v \cdot \mathbf{N} + \alpha_{11} \mathbf{N} \cdot \mathbf{N} = \alpha_{11} \\ M &= \mathbf{x}_{uv} \cdot \mathbf{N} = \Gamma_{12}^1 \mathbf{x}_u \cdot \mathbf{N} + \Gamma_{12}^2 \mathbf{x}_v \cdot \mathbf{N} + \alpha_{12} \mathbf{N} \cdot \mathbf{N} = \alpha_{12} \\ N &= \mathbf{x}_{vv} \cdot \mathbf{N} = \Gamma_{22}^1 \mathbf{x}_u \cdot \mathbf{N} + \Gamma_{22}^2 \mathbf{x}_v \cdot \mathbf{N} + \alpha_{22} \mathbf{N} \cdot \mathbf{N} = \alpha_{22}\end{aligned}\quad (10.3)$$

Thus

It remains to determine the Γ_{ij}^k .

In Problem 10.3, page 216, we prove that they are given by

$$\begin{aligned}\Gamma_{11}^1 &= \frac{GE_u - 2FF_u + FE_v}{2(EG - F^2)} & \Gamma_{12}^1 &= \frac{GE_v - FG_u}{2(EG - F^2)} & \Gamma_{22}^1 &= \frac{2GF_v - GG_u - FG_v}{2(EG - F^2)} \\ \Gamma_{11}^2 &= \frac{2EF_u - EE_v + FE_v}{2(EG - F^2)} & \Gamma_{12}^2 &= \frac{EG_u - FE_v}{2(EG - F^2)} & \Gamma_{22}^2 &= \frac{EG_v - 2FF_v + FG_u}{2(EG - F^2)}\end{aligned}\quad (10.4)$$

Thus we have

Theorem 10.1. On a patch $\mathbf{x} = \mathbf{x}(u, v)$ on a surface of class ≥ 2 the vectors $\mathbf{x}_u, \mathbf{x}_v, \mathbf{N}$ and their derivatives satisfy

$$\begin{aligned}\mathbf{x}_{uu} &= \Gamma_{11}^1 \mathbf{x}_u + \Gamma_{11}^2 \mathbf{x}_v + L\mathbf{N} \\ \mathbf{x}_{uv} &= \Gamma_{12}^1 \mathbf{x}_u + \Gamma_{12}^2 \mathbf{x}_v + M\mathbf{N} \\ \mathbf{x}_{vv} &= \Gamma_{22}^1 \mathbf{x}_u + \Gamma_{22}^2 \mathbf{x}_v + N\mathbf{N} \\ \mathbf{N}_u &= \beta_1^1 \mathbf{x}_u + \beta_1^2 \mathbf{x}_v \\ \mathbf{N}_v &= \beta_2^1 \mathbf{x}_u + \beta_2^2 \mathbf{x}_v\end{aligned}\quad (10.5)$$

where the coefficients β_i^k and Γ_{ij}^k are given by equations (10.2) and (10.4).

The first three of the above equations are called the *Gauss* equations and the last two the *Weingarten* equations. The quantities Γ_{ij}^k are called the *Christoffel symbols* of the second kind. Observe in (10.4) that the Γ_{ij}^k depend only upon the first fundamental coefficients and their derivatives as compared to the β_i^k which depend on both the first and second fundamental coefficients. We also define $\Gamma_{21}^1 = \Gamma_{12}^1$ and $\Gamma_{21}^2 = \Gamma_{12}^2$. Thus $\Gamma_{ij}^k = \Gamma_{ji}^k$ for all $i, j, k = 1, 2$.

Example 10.1.

We want to verify equations (10.2) and (10.4) for the surface of revolution

$$\mathbf{x} = (u \cos \theta) \mathbf{e}_1 + (u \sin \theta) \mathbf{e}_2 + g(u) \mathbf{e}_3, \quad u > 0$$

Here

$$\mathbf{x}_u = (\cos \theta) \mathbf{e}_1 + (\sin \theta) \mathbf{e}_2 + g'(u) \mathbf{e}_3, \quad \mathbf{x}_\theta = -(u \sin \theta) \mathbf{e}_1 + (u \cos \theta) \mathbf{e}_2$$

$$\mathbf{N} = -(1 + g'^2)^{-1/2} (g'(\cos \theta) \mathbf{e}_1 + g'(\sin \theta) \mathbf{e}_2 - \mathbf{e}_3)$$

$$\mathbf{x}_{uu} = g'' \mathbf{e}_3, \quad \mathbf{x}_{u\theta} = -(\sin \theta) \mathbf{e}_1 + (\cos \theta) \mathbf{e}_2, \quad \mathbf{x}_{\theta\theta} = -(u \cos \theta) \mathbf{e}_1 - (u \sin \theta) \mathbf{e}_2$$

$$E = \mathbf{x}_u \cdot \mathbf{x}_u = 1 + g'^2, \quad F = \mathbf{x}_u \cdot \mathbf{x}_\theta = 0, \quad G = \mathbf{x}_\theta \cdot \mathbf{x}_\theta = u^2$$

$$L = \mathbf{x}_{uu} \cdot \mathbf{N} = g''(1 + g'^2)^{-1/2}, \quad M = \mathbf{x}_{u\theta} \cdot \mathbf{N} = 0, \quad N = \mathbf{x}_{\theta\theta} \cdot \mathbf{N} = ug'(1 + g'^2)^{-1/2}$$

Using equations (10.2) and (10.4) it can be computed that

$$\beta_1^1 = -g''(1 + g'^2)^{-3/2}, \quad \beta_1^2 = \beta_2^1 = 0, \quad \beta_2^2 = -u^{-1}g'(1 + g'^2)^{-1/2}$$

$$\Gamma_{11}^1 = g'g''/(1 + g'^2), \quad \Gamma_{11}^2 = \Gamma_{12}^1 = \Gamma_{22}^2 = 0, \quad \Gamma_{12}^2 = u^{-1}, \quad \Gamma_{22}^1 = -u/(1 + g'^2)$$

It follows that

$$\Gamma_{11}^1 \mathbf{x}_u + \Gamma_{11}^2 \mathbf{x}_\theta + L\mathbf{N} = g'g''/(1 + g'^2) \mathbf{x}_u + g''(1 + g'^2)^{-1/2} \mathbf{N} = g'' \mathbf{e}_3 = \mathbf{x}_{uu}$$

$$\Gamma_{12}^1 \mathbf{x}_u + \Gamma_{12}^2 \mathbf{x}_\theta + M\mathbf{N} = u^{-1} \mathbf{x}_\theta = -(\sin \theta) \mathbf{e}_1 + (\cos \theta) \mathbf{e}_2 = \mathbf{x}_{u\theta}$$

$$\Gamma_{22}^1 \mathbf{x}_u + \Gamma_{22}^2 \mathbf{x}_\theta + N\mathbf{N} = -u/(1 + g'^2) \mathbf{x}_u + ug'(1 + g'^2)^{-1/2} \mathbf{N} = -(u \cos \theta) \mathbf{e}_1 - (u \sin \theta) \mathbf{e}_2 = \mathbf{x}_{\theta\theta}$$

$$\beta_1^1 \mathbf{x}_u + \beta_1^2 \mathbf{x}_\theta = -g''(1 + g'^2)^{-3/2} \mathbf{x}_u = -g''(1 + g'^2)^{-3/2} ((\cos \theta) \mathbf{e}_1 + (\sin \theta) \mathbf{e}_2 + g' \mathbf{e}_3) = \mathbf{N}_u$$

$$\beta_2^1 \mathbf{x}_u + \beta_2^2 \mathbf{x}_\theta = -u^{-1}g'(1 + g'^2)^{-1/2} \mathbf{x}_\theta = g'(1 + g'^2)^{-1/2} ((\sin \theta) \mathbf{e}_1 - (\cos \theta) \mathbf{e}_2) = \mathbf{N}_\theta$$

which is the required result.

THE COMPATIBILITY EQUATIONS AND THE THEOREM OF GAUSS

Given functions E, F, G, L, M and N of u and v of sufficiently high class, we want to investigate whether or not there exists a surface $\mathbf{x} = \mathbf{x}(u, v)$ for which E, F, G, L, M and N are the first and second fundamental coefficients. In general, the answer is in the negative

unless certain "compatibility" conditions are satisfied. These conditions arise from the fact that if $\mathbf{x}(u, v)$ is a function of class C^3 , then the third order mixed partial derivatives of $\mathbf{x}$ are independent of the order of differentiation; i.e.,

$$(\mathbf{x}_u)_{uv} = (\mathbf{x}_v)_{vu}, \quad (\mathbf{x}_v)_{uv} = (\mathbf{x}_u)_{vu} \quad (10.6)$$

In Problem 10.28, page 224, we prove

Theorem 10.2. Let $\mathbf{x} = \mathbf{x}(u, v)$ be a patch on a surface of class ≥ 2 such that the coefficients of the Gauss-Weingarten equations are of class C^1 . Then the mixed derivatives $\mathbf{x}_{uuv}$, $\mathbf{x}_{uvu}$, $\mathbf{x}_{uvv}$, $\mathbf{x}_{vuv}$, $\mathbf{x}_{vvu}$ exist and satisfy (10.6) above if and only if the first and second fundamental coefficients satisfy the *compatibility* equations

$$L_v - M_u = L\Gamma_{12}^1 + M(\Gamma_{12}^2 - \Gamma_{11}^1) - N\Gamma_{11}^2 \quad (10.7)$$

$$M_v - N_u = L\Gamma_{22}^1 + M(\Gamma_{22}^2 - \Gamma_{12}^1) - N\Gamma_{12}^2$$

and

$$LN - M^2 = F[(\Gamma_{22}^2)_u - (\Gamma_{12}^2)_v + \Gamma_{22}^1\Gamma_{11}^2 - \Gamma_{12}^1\Gamma_{12}^2] + E[(\Gamma_{22}^1)_u - (\Gamma_{12}^1)_v + \Gamma_{22}^1\Gamma_{11}^1 + \Gamma_{22}^2\Gamma_{12}^1 - \Gamma_{12}^1\Gamma_{12}^2 - \Gamma_{12}^2\Gamma_{22}^1] \quad (10.8)$$

The compatibility equations can be written in various analytical forms. In this form the first two equations (10.7) are called the Mainardi-Codazzi equations.

Equation (10.8) is of particular interest. We recall that the Γ_{ij}^k depend only upon the first fundamental coefficients and their derivatives. Hence $LN - M^2$ depends only on E, F, G and their derivatives. But then the Gaussian curvature $K = (LN - M^2)/(EG - F^2)$, which was originally defined in terms of the second fundamental form, *depends only upon the coefficients of the first fundamental form*. This is one of the most important results of the theory of surfaces and we will see that it has many important consequences. Thus we have

Theorem 10.3. **Theorema Egregium of Gauss.** The Gaussian curvature on a surface of class ≥ 3 is a function only of the coefficients of the first fundamental form (and their derivatives).

THE FUNDAMENTAL THEOREM OF SURFACES

Theorem 10.4. **Fundamental Theorem of Surfaces.** Let E, F and G be functions of u and v of class C^2 and let L, M and N be functions of u and v of class C^1 all defined on an open set containing (u_0, v_0) such that for all (u, v) ,

- (i) $EG - F^2 > 0$, $E > 0$, $G > 0$
- (ii) E, F, G, L, M, N satisfy the compatibility equations (10.7) and (10.8).

Then there exists a patch $\mathbf{x} = \mathbf{x}(u, v)$ of class C^3 defined in a neighborhood of (u_0, v_0) for which E, F, G, L, M, N are the first and second fundamental coefficients. The surface represented by $\mathbf{x} = \mathbf{x}(u, v)$ is unique except for position in space.

A proof of the existence of the surface with the given functions E, F, G, L, M, N as first and second fundamental coefficients is given in Appendix 2. We now prove uniqueness. We suppose there are two patches, $\mathbf{x} = \mathbf{x}(u, v)$ and $\mathbf{x} = \mathbf{x}^*(u, v)$, defined in an open connected set U containing (u_0, v_0) and such that for all (u, v) the coefficients $E = E^*$, $F = F^*$, $G = G^*$, $L = L^*$, $M = M^*$ and $N = N^*$. We suppose that the surface represented by $\mathbf{x} = \mathbf{x}^*(u, v)$ is translated and then rotated so that the point corresponding to $\mathbf{x}^*(u_0, v_0)$ coincides with $\mathbf{x}(u_0, v_0)$ and that tangent vectors $\mathbf{x}_u^*(u_0, v_0)$ and $\mathbf{x}_v^*(u_0, v_0)$ coincide with $\mathbf{x}_u(u_0, v_0)$ and $\mathbf{x}_v(u_0, v_0)$ respectively. This is possible since the lengths of the vectors $\mathbf{x}_u^*$ and $\mathbf{x}_v^*$ and the angle between them are determined by E^* , F^* and G^* which agree with

E, F and G at (u_0, v_0) . Now let $u = u(t), v = v(t)$ be a regular arc connecting (u_0, v_0) with an arbitrary point (u, v) in U , and consider the functions $\mathbf{x}(t) = \mathbf{x}(u(t), v(t))$, $\mathbf{x}_u(t) = \mathbf{x}_u(u(t), v(t))$ and $\mathbf{x}_v(t) = \mathbf{x}_v(u(t), v(t))$. Differentiating,

$$\frac{d\mathbf{x}}{dt} = \mathbf{x}_u \frac{du}{dt} + \mathbf{x}_v \frac{dv}{dt}, \quad \frac{d\mathbf{x}_u}{dt} = \mathbf{x}_{uu} \frac{du}{dt} + \mathbf{x}_{uv} \frac{dv}{dt}, \quad \frac{d\mathbf{x}_v}{dt} = \mathbf{x}_{vu} \frac{du}{dt} + \mathbf{x}_{vv} \frac{dv}{dt}$$

Using the first three equations of (10.5) and the fact that $\mathbf{N} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{|\mathbf{x}_u \times \mathbf{x}_v|} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\sqrt{EG - F^2}}$, we obtain

$$\begin{aligned} \frac{d\mathbf{x}}{dt} &= \mathbf{x}_u \frac{du}{dt} + \mathbf{x}_v \frac{dv}{dt} \\ \frac{d\mathbf{x}_u}{dt} &= \left(\Gamma_{11}^1 \frac{du}{dt} + \Gamma_{12}^1 \frac{dv}{dt} \right) \mathbf{x}_u + \left(\Gamma_{11}^2 \frac{du}{dt} + \Gamma_{12}^2 \frac{dv}{dt} \right) \mathbf{x}_v \\ &\quad + \left(L \frac{du}{dt} + M \frac{dv}{dt} \right) (\mathbf{x}_u \times \mathbf{x}_v) (EG - F^2)^{-1/2} \\ \frac{d\mathbf{x}_v}{dt} &= \left(\Gamma_{12}^1 \frac{du}{dt} + \Gamma_{22}^1 \frac{dv}{dt} \right) \mathbf{x}_u + \left(\Gamma_{12}^2 \frac{du}{dt} + \Gamma_{22}^2 \frac{dv}{dt} \right) \mathbf{x}_v \\ &\quad + \left(M \frac{du}{dt} + N \frac{dv}{dt} \right) (\mathbf{x}_u \times \mathbf{x}_v) (EG - F^2)^{-1/2} \end{aligned}$$

or

$$\begin{aligned} \frac{d\mathbf{x}}{dt} &= a(t)\mathbf{x}_u + b(t)\mathbf{x}_v \\ \frac{d\mathbf{x}_u}{dt} &= c(t)\mathbf{x}_u + d(t)\mathbf{x}_v + e(t)\mathbf{x}_u \times \mathbf{x}_v \\ \frac{d\mathbf{x}_v}{dt} &= f(t)\mathbf{x}_u + g(t)\mathbf{x}_v + h(t)\mathbf{x}_u \times \mathbf{x}_v \end{aligned} \quad (10.9)$$

where

$$a(t) = \frac{du}{dt}, \quad b(t) = \frac{dv}{dt}, \quad c(t) = \Gamma_{11}^1 \frac{du}{dt} + \Gamma_{12}^1 \frac{dv}{dt}, \quad \text{etc.}$$

We now regard the above equations as a system of first order ordinary differential equations for the functions $\mathbf{x}(t)$, $\mathbf{x}_u(t)$ and $\mathbf{x}_v(t)$. We note also that the coefficients $a(t)$, $b(t)$, etc., depend only upon the curve $u(t), v(t)$ and the first and second fundamental coefficients E, F, G, L, M, N and the derivatives of E, F and G along $u = u(t), v = v(t)$. Since $E = E^*, F = F^*, G = G^*, L = L^*, M = M^*$ and $N = N^*$ for all (u, v) , the corresponding functions $\mathbf{x}^*(t) = \mathbf{x}^*(u(t), v(t))$, $\mathbf{x}_u^*(t) = \mathbf{x}_u^*(u(t), v(t))$, and $\mathbf{x}_v^*(t) = \mathbf{x}_v^*(u(t), v(t))$ along the patch $\mathbf{x} = \mathbf{x}^*(u, v)$ satisfy the same system of equations (10.9). Also initially $\mathbf{x}(t_0) = \mathbf{x}(u_0, v_0) = \mathbf{x}^*(u_0, v_0) = \mathbf{x}^*(t_0)$, $\mathbf{x}_u(t_0) = \mathbf{x}_u(u_0, v_0) = \mathbf{x}_u^*(u_0, v_0) = \mathbf{x}_u^*(t_0)$ and $\mathbf{x}_v(t_0) = \mathbf{x}_v(u_0, v_0) = \mathbf{x}_v^*(u_0, v_0) = \mathbf{x}_v^*(t_0)$. It follows from the uniqueness theorem for ordinary differential equations that $\mathbf{x}(t) = \mathbf{x}^*(t)$ everywhere along the curve $u = u(t), v = v(t)$. Hence the patches $\mathbf{x} = \mathbf{x}(u, v)$ and $\mathbf{x} = \mathbf{x}^*(u, v)$ coincide, which proves the theorem.

Example 10.2.

We want to determine the surface whose fundamental coefficients are $E = 1, F = 0, G = \sin^2 u, L = 1, M = 0$ and $N = \sin^2 u, 0 < u < \pi$. From equations (10.2) and (10.4) we obtain $\beta_1^1 = -1, \beta_2^1 = \beta_2^2 = 0, \beta_2^2 = -1, \Gamma_{11}^1 = \Gamma_{11}^2 = \Gamma_{12}^1 = \Gamma_{12}^2 = 0, \Gamma_{12}^2 = \cot u, \Gamma_{22}^1 = -\sin u \cos u$. Thus the Gauss-Weingarten equations are

$$\mathbf{x}_{uu} = \mathbf{N}, \quad \mathbf{x}_{uv} = (\cot u)\mathbf{x}_v, \quad \mathbf{x}_{vv} = -(\sin u \cos u)\mathbf{x}_u + (\sin^2 u)\mathbf{N}, \quad \mathbf{N}_u = -\mathbf{x}_u, \quad \mathbf{N}_v = -\mathbf{x}_v$$

From the first and fourth equations we obtain $\mathbf{x}_{uuu} = -\mathbf{x}_u$. Integrating gives

$$\mathbf{x} = a(v) \sin u + b(v) \cos u + c(v)$$

From the second equation we obtain

$$\mathbf{x}_{uv} = a' \cos u - b' \sin u = (\cot u)\mathbf{x}_v = a' \cos u + b' \cos u \cot u + c' \cot u$$

Thus $b'(\sin u + \cos u \cot u) = -c' \cot u$ or $b' = -c' \cos u$. Since b' and c' are functions only of v , it follows that $b' = c' = 0$. Hence $b = \text{constant}$ and $c = \text{constant}$. From the first and third equations we obtain further that

$$\mathbf{x}_{vv} = a'' \sin u = -(\sin u \cos u)\mathbf{x}_u + (\sin^2 u)\mathbf{x}_{uu} = -a \sin u$$

Hence $\mathbf{a}'' = -\mathbf{a}$ and $\mathbf{a} = \mathbf{d} \cos v + \mathbf{e} \sin v$, where $\mathbf{d} = \text{constant}$ and $\mathbf{e} = \text{constant}$. Thus

it remains to show that the vectors $\mathbf{d}, \mathbf{e}$ and $\mathbf{b}$ form an orthonormal set. Thus

$$\begin{aligned} |\mathbf{x} - \mathbf{c}|^2 &= (\mathbf{d} \cdot \mathbf{d}) \cos^2 v \sin^2 u + 2(\mathbf{d} \cdot \mathbf{e}) \cos v \sin v \sin^2 u \\ &\quad + 2(\mathbf{d} \cdot \mathbf{b}) \cos v \sin u \cos u + (\mathbf{e} \cdot \mathbf{e}) \sin^2 v \sin^2 u \\ &\quad + 2(\mathbf{e} \cdot \mathbf{b}) \sin v \sin u \cos u + (\mathbf{b} \cdot \mathbf{b}) \cos^2 v \sin^2 u \\ &= \cos^2 v \sin^2 u + \sin^2 v \sin^2 u + \cos^2 u = 1 \end{aligned}$$

That is, $\mathbf{x}$ is on a sphere of radius 1 about $\mathbf{c}$. In order to show that $\mathbf{d}, \mathbf{e}$ and $\mathbf{b}$ form an orthonormal set, we observe that

$$\mathbf{x}_v \cdot \mathbf{x}_v = G = \sin^2 u = (\mathbf{d} \cdot \mathbf{d}) \sin^2 v \sin^2 u - 2(\mathbf{d} \cdot \mathbf{e}) \sin v \cos v \sin^2 u + (\mathbf{e} \cdot \mathbf{e}) \cos^2 v \sin^2 u$$

or

$$1 = (\mathbf{d} \cdot \mathbf{d}) \sin^2 v - 2(\mathbf{d} \cdot \mathbf{e}) \sin v \cos v + (\mathbf{e} \cdot \mathbf{e}) \cos^2 v$$

Hence $\mathbf{e} \cdot \mathbf{e} = 1$, $\mathbf{d} \cdot \mathbf{e} = 0$, $\mathbf{d} \cdot \mathbf{d} = 1$. Using this, we obtain further that

$$\mathbf{x}_v \cdot \mathbf{x}_u = F = 0 = (\mathbf{b} \cdot \mathbf{d}) \sin v \sin^2 u - (\mathbf{e} \cdot \mathbf{b}) \cos v \sin^2 u$$

Hence $\mathbf{b} \cdot \mathbf{d} = \mathbf{e} \cdot \mathbf{b} = 0$. Finally

$$\mathbf{x}_u \cdot \mathbf{x}_u = E = 1 = \cos^2 v \cos^2 u + \sin^2 v \cos^2 u + (\mathbf{b} \cdot \mathbf{b}) \sin^2 u$$

Thus $\mathbf{b} \cdot \mathbf{b} = 1$, which is the required result. Accordingly the surface with the given fundamental coefficients is a sphere of radius one.

SOME THEOREMS ON SURFACES IN THE LARGE

We want to prove that the sphere is the only connected and closed surface of class ≥ 3 all points of which are spherical umbilical points. Suppose then that S is a surface of class ≥ 3 which is connected, closed and that every point on S is a spherical umbilical point. Let P be an arbitrary point on S and $\mathbf{x} = \mathbf{x}(u, v)$ a connected patch on S containing P . We recall that at a spherical umbilical point the normal curvature $\kappa = \text{constant} \neq 0$ in every direction and so every direction on the patch is a principal direction. Thus every curve on the patch and in particular the parameter curves are lines of curvature. It follows from Rodrigues' formula that $\mathbf{N}_u = -\kappa \mathbf{x}_u$ and $\mathbf{N}_v = -\kappa \mathbf{x}_v$. Note that at a fixed point κ is constant in every direction but it is not known a priori that κ is constant from point to point on the patch. To prove this we use the fact that $\mathbf{x}(u, v)$ is of class C^3 and compute $\mathbf{N}_{uv} = -\kappa \mathbf{x}_{uv} - \kappa_v \mathbf{x}_u$ and $\mathbf{N}_{vu} = -\kappa \mathbf{x}_{vu} - \kappa_u \mathbf{x}_v$. Subtracting gives $\kappa_v \mathbf{x}_u - \kappa_u \mathbf{x}_v = 0$. But at each point $\mathbf{x}_u$ and $\mathbf{x}_v$ are linearly independent. Thus $\kappa_v = 0$ and $\kappa_u = 0$. Hence $\kappa = \text{constant}$ on the patch. Thus every point on S belongs to a patch on which $\kappa = \text{constant} \neq 0$. Now consider P to be fixed and Q to be any other point on S . Since S is connected, there exists a regular arc $\Gamma: \mathbf{x} = \mathbf{x}(t)$ on S connecting P and Q . Since every point on S is a spherical umbilical point, every curve on S is a line of curvature. Thus again from Rodrigues' formula along Γ , $d\mathbf{N}/dt = -\kappa(d\mathbf{x}/dt)$ where we note that $\kappa = \text{constant}$ since at each point on Γ , $\kappa = \text{constant}$ on a patch containing the point. It follows by integrating that

$$\mathbf{N} = -\kappa \mathbf{x} + \mathbf{C}, \quad \text{or} \quad |\mathbf{x} - \mathbf{C}/\kappa| = 1/|\kappa| \quad (\mathbf{C} = \text{constant})$$

Thus $\mathbf{x}$ and in particular Q is on the sphere Σ of radius $1/|\kappa|$ with center at $\mathbf{C}/\kappa$. Since Q is arbitrary it follows that S is on Σ . Since Σ is connected and S is closed it follows from Theorem 8.5, page 159, that $S = \Sigma$; this proves

Theorem 10.5. The only connected and closed surfaces of class ≥ 3 of which all points are spherical umbilical points are spheres.

Similarly one can prove

Theorem 10.6. The only connected and closed surfaces of class ≥ 2 of which all points are planar points are planes.

The proof of the above theorem is left to the reader as an exercise (Problem 10.38).

In Problem 10.10 we also prove

Theorem 10.7. (Liebmann). The only connected and compact surfaces of sufficiently high class with constant Gaussian curvature are spheres.

We note that as a result of the above theorem we have a particularly important property of spheres which will be considered in greater detail in the next chapter. Namely, suppose there exists a continuous mapping $\mathbf{f}$ of a surface Σ onto a surface S which is locally 1-1 and preserves the first fundamental form, i.e. for each point P on Σ there exists a patch $\mathbf{x} = \mathbf{x}(u, v)$ containing P such that $\mathbf{f}$ is a 1-1 mapping of $\mathbf{x} = \mathbf{x}(u, v)$ onto a patch $\mathbf{x} = \mathbf{x}^*(u, v)$ on S and such that first fundamental coefficients agree at corresponding points. Note that as shown in Problem 9.6, page 190, the two surfaces need not be the same since a surface is uniquely determined by both its first and second fundamental coefficients. However, if Σ is a sphere then S must be a sphere of the same radius. For a sphere Σ has constant Gaussian curvature equal to $K = 1/R^2$, where R is the radius of Σ , and since the Gaussian curvature is a function only of the first fundamental coefficients it follows that S also has constant Gaussian curvature equal to $K = 1/R^2$. Also since the sphere Σ is connected and compact and the mapping $\mathbf{f}$ of Σ onto S is continuous, it follows that S is connected and compact. But then it follows from the above theorem that S is also a sphere and, moreover, its radius is $R = 1/K^{1/2}$.

NOTATION

The formalism of the theory of surfaces can be greatly simplified with the use of tensors and tensor notation. However, this will require a change in notation. The components of a vector will be denoted by superscripts in place of subscripts. Thus a vector in E^3 is $\mathbf{x} = x^1\mathbf{e}_1 + x^2\mathbf{e}_2 + x^3\mathbf{e}_3$, a point in the parameter plane is (u^1, u^2) , and a patch is $\mathbf{x} = \mathbf{x}(u^1, u^2)$.

Also the partial derivatives of $\mathbf{x}$ shall be denoted by

$$\mathbf{x}_1 = \frac{\partial \mathbf{x}}{\partial u^1}, \quad \mathbf{x}_2 = \frac{\partial \mathbf{x}}{\partial u^2}, \quad \mathbf{x}_{12} = \frac{\partial^2 \mathbf{x}}{\partial u^1 \partial u^2}, \quad \mathbf{x}_{22} = \frac{\partial^2 \mathbf{x}}{\partial u^2 \partial u^2}, \quad \text{etc.}$$

It follows that a tangent vector is $d\mathbf{x} = \mathbf{x}_1 du^1 + \mathbf{x}_2 du^2$ and the first fundamental form is

$$\begin{aligned} I &= d\mathbf{x} \cdot d\mathbf{x} = \mathbf{x}_1 \cdot \mathbf{x}_1 du^1 du^1 + 2\mathbf{x}_1 \cdot \mathbf{x}_2 du^1 du^2 + \mathbf{x}_2 \cdot \mathbf{x}_2 du^2 du^2 \\ &= g_{11} du^1 du^1 + g_{12} du^1 du^2 + g_{21} du^2 du^1 + g_{22} du^2 du^2 = \sum_{ik} g_{ik} du^i du^k \end{aligned} \quad (10.10)$$

where $g_{11} = \mathbf{x}_1 \cdot \mathbf{x}_1 = E$, $g_{12} = g_{21} = \mathbf{x}_1 \cdot \mathbf{x}_2 = F$ and $g_{22} = \mathbf{x}_2 \cdot \mathbf{x}_2 = G$ denote the first fundamental coefficients and $i, k = 1, 2$.

We also use g to denote the discriminant of I ; i.e.,

$$g = \det \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = g_{11}g_{22} - g_{21}g_{12} = EG - F^2$$

and introduce the quantities

$$g^{11} = g_{22}/g, \quad g^{12} = g^{21} = -g_{12}/g = -g_{21}/g, \quad g^{22} = g_{11}/g \quad (10.11)$$

In Problem 10.12, page 220, we show that

$$\sum_k g_{ik} g^{kj} = \delta_i^j = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{otherwise} \end{cases}$$

That is, the matrix (g^{ij}) is the inverse of (g_{ij}) , and the product

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (10.12)$$

Finally, the differential of the normal is the vector $d\mathbf{N} = N_1 du^1 + N_2 du^2$ and the second fundamental form is

$$\Pi = -d\mathbf{x} \cdot d\mathbf{N} = -\mathbf{x}_1 \cdot N_1 du^1 du^1 - \mathbf{x}_1 \cdot N_2 du^1 du^2 - \mathbf{x}_2 \cdot N_1 du^2 du^1 - \mathbf{x}_2 \cdot N_2 du^2 du^2$$

where

$$b_{11} = -\mathbf{x}_1 \cdot N_1 = \mathbf{x}_{11} \cdot \mathbf{N} = L$$

$$b_{12} = b_{21} = -\mathbf{x}_2 \cdot N_1 = -\mathbf{x}_1 \cdot N_2 = \mathbf{x}_{12} \cdot \mathbf{N} = M$$

$$b_{22} = -\mathbf{x}_2 \cdot N_2 = \mathbf{x}_{22} \cdot \mathbf{N} = N$$

now denote the second fundamental coefficients. We also define

$$b = \det(b_{ij}) = (b_{11}b_{22} - b_{12}b_{21}) = LN - M^2 \quad (10.15)$$

The summation convention to be used is as follows: Consider the sum

$$\sum_{\alpha=1}^3 a_{\alpha} b^{\alpha} = a_{11}b^1 + a_{12}b^2 + a_{13}b^3$$

Observe on the left that α appears in the product $a_{\alpha}b^{\alpha}$ exactly once as a subscript and once as a superscript. When this is the case we omit the Σ sign and write simply $a_{\alpha}b^{\alpha}$. Thus

$$a_{\alpha}b^{\alpha} = \sum_{\alpha} a_{\alpha}b^{\alpha} = a_{11}b^1 + a_{12}b^2 + a_{13}b^3$$

The index α which is summed over is called a *summation* or *dummy* index. A dummy index can always be changed in a computation. That is,

$$a_{\alpha}b^{\alpha} = a_{\beta}b^{\beta} = a_{\gamma}b^{\gamma}$$

The index i is called a *free* index. It cannot be changed. Finally in a derivative $\partial\theta/\partial u^i$ or $\mathbf{x}_j = \partial\mathbf{x}/\partial u^j$, the index i is considered a superscript and j a subscript.

Example 10.3.

(a) Let $f = f(x^1, x^2, x^3)$ and $x^i = x^i(u^1, u^2)$, $i = 1, 2, 3$. The chain rule for the derivatives of f with respect to the u^i is

$$\frac{\partial f}{\partial u^i} = \frac{\partial f}{\partial x^1} \frac{\partial x^1}{\partial u^i} + \frac{\partial f}{\partial x^2} \frac{\partial x^2}{\partial u^i} + \frac{\partial f}{\partial x^3} \frac{\partial x^3}{\partial u^i} = \sum_{\alpha=1}^3 \frac{\partial f}{\partial x^{\alpha}} \frac{\partial x^{\alpha}}{\partial u^i}, \quad i = 1, 2$$

Since α occurs as "subscript" in $\frac{\partial f}{\partial x^{\alpha}}$ and as a "superscript" in $\frac{\partial x^{\alpha}}{\partial u^i}$, this can be written as

$$\frac{\partial f}{\partial u^i} = \frac{\partial f}{\partial x^{\alpha}} \frac{\partial x^{\alpha}}{\partial u^i}$$

(b) Let $S = a_{\alpha\beta}x^{\alpha}x^{\beta}$, $\alpha, \beta = 1, 2, 3$. Since both α and β occur once as a subscript and once as a superscript, S is the double sum

$$\begin{aligned} S &= \sum_{\alpha=1}^3 \sum_{\beta=1}^3 a_{\alpha\beta}x^{\alpha}x^{\beta} \\ &= a_{11}x^1x^1 + a_{12}x^1x^2 + a_{13}x^1x^3 + a_{21}x^2x^1 + a_{22}x^2x^2 + a_{23}x^2x^3 + a_{31}x^3x^1 + a_{32}x^3x^2 + a_{33}x^3x^3 \end{aligned}$$

Example 10.4.

Using the summation convention, we can write a tangent vector as $d\mathbf{x} = \mathbf{x}_{\alpha} du^{\alpha}$ and the first fundamental form as $I = \mathbf{x}_{\alpha} \cdot \mathbf{x}_{\beta} du^{\alpha} du^{\beta}$. The differential of the normal is $d\mathbf{N} = N_{\alpha} du^{\alpha}$ and the second fundamental form is $\Pi = -\mathbf{x}_{\alpha} \cdot N_{\beta} du^{\alpha} du^{\beta} = \mathbf{x}_{\alpha\beta} \cdot \mathbf{N} du^{\alpha} du^{\beta} = b_{\alpha\beta} du^{\alpha} du^{\beta}$. The Gauss-Weingarten equations (10.5) are written as

$$\begin{aligned} \mathbf{x}_{11} &= \Gamma_{11}^{\alpha} \mathbf{x}_{\alpha} + b_{11} \mathbf{N}, & \mathbf{x}_{12} &= \Gamma_{12}^{\alpha} \mathbf{x}_{\alpha} + b_{12} \mathbf{N}, & \mathbf{x}_{22} &= \Gamma_{22}^{\alpha} \mathbf{x}_{\alpha} + b_{22} \mathbf{N}, & N_1 &= \beta_1^{\alpha} \mathbf{x}_{\alpha}, & N_2 &= \beta_2^{\alpha} \mathbf{x}_{\alpha} \\ & & & & & & & & & & & i, j = 1, 2 \\ & & & & & & & & & & & \mathbf{x}_{ij} &= \Gamma_{ij}^{\alpha} \mathbf{x}_{\alpha} + b_{ij} \mathbf{N}, & N_i &= \beta_i^{\alpha} \mathbf{x}_{\alpha}, \end{aligned}$$

or simply

ELEMENTARY MANIFOLDS

Since tensors have a wide variety of applications in addition to surfaces in E^3 , we will introduce a generalization of the concept of an elementary surface as follows: We assume we have an abstract collection M of objects P called "points" which can be put into a 1-1 correspondence with a set S of n -tuples of real numbers $(u^1, u^2, \dots, u^n)$ called the coordinates of P . This correspondence $P(u^1, u^2, \dots, u^n)$ between the points in M and the set of n -tuples S is called a *coordinate system* on M . A coordinate system $P(u^1, \dots, u^n)$ on M is analogous to a patch $x = x(u^1, u^2)$ on a surface, which determines a 1-1 correspondence with the points on the patch and a set of points in the $u^1 u^2$ plane. Any other coordinate system $P(\bar{u}^1, \dots, \bar{u}^n)$ on M defined on a set of n -tuples $\bar{S}$ will determine a 1-1 correspondence $(u^1, \dots, u^n) \leftrightarrow (\bar{u}^1, \dots, \bar{u}^n)$ between the two sets of n -tuples S and $\bar{S}$ called a *coordinate transformation*. This correspondence can be written as $\bar{u}^i = \bar{u}^i(u^1, \dots, u^n)$, $i = 1, \dots, n$, with inverse equations $u^i = u^i(\bar{u}^1, \dots, \bar{u}^n)$. Here the $\bar{u}^i$ are real valued functions on S and the u^i are real valued functions on $\bar{S}$. These equations correspond to the parameter transformation $\bar{u}^1 = \bar{u}^1(u^1, u^2)$, $\bar{u}^2 = \bar{u}^2(u^1, u^2)$ with inverse $u^1 = u^1(\bar{u}^1, \bar{u}^2)$, $u^2 = u^2(\bar{u}^1, \bar{u}^2)$ which exist on the intersection of two patches on a surface.

We assume that the sets S of n -tuples of real numbers on which the coordinate systems are defined are *open*. A set S of n -tuples of real numbers $(u^1, \dots, u^n)$ is open if for every $(u_0^1, \dots, u_0^n)$ in S there exists a real $\epsilon > 0$ such that all $(u^1, \dots, u^n)$ satisfying $\left[\sum_{i=1}^n (u^i - u_0^i)^2 \right]^{1/2} < \epsilon$ are in S .

We also assume that the transformations $\bar{u}^i = \bar{u}^i(u^1, \dots, u^n)$ and their inverses $u^i = u^i(\bar{u}^1, \dots, \bar{u}^n)$ have continuous partial derivatives $\frac{\partial \bar{u}^i}{\partial u^j}$ and $\frac{\partial u^i}{\partial \bar{u}^j}$, $i, j = 1, \dots, n$, and non-vanishing Jacobians $\frac{\partial(\bar{u}^1, \dots, \bar{u}^n)}{\partial(u^1, \dots, u^n)} = \det\left(\frac{\partial \bar{u}^i}{\partial u^j}\right) \neq 0$ and $\frac{\partial(u^1, \dots, u^n)}{\partial(\bar{u}^1, \dots, \bar{u}^n)} = \det\left(\frac{\partial u^i}{\partial \bar{u}^j}\right) \neq 0$. In Problem 10.17, page 221, we prove that since the $\bar{u}_i$ and u_i are functions which are inverse to each other, we have

$$\frac{\partial \bar{u}^i}{\partial u^\alpha} \frac{\partial u^\alpha}{\partial \bar{u}^i} = \delta_i^i \quad (10.16)$$

A real valued function $\bar{u}^i(u^1, \dots, u^n)$ defined on S is continuous at $(u_0^1, \dots, u_0^n)$ if given an $\epsilon > 0$ there exists a $\delta > 0$ such that $\left[\sum_i (\bar{u}^i(u^1, \dots, u^n) - \bar{u}^i(u_0^1, \dots, u_0^n))^2 \right]^{1/2} < \epsilon$ for $\left[\sum_i (u^i - u_0^i)^2 \right]^{1/2} < \delta$. The function $\bar{u}^i(u^1, \dots, u^n)$ is continuous on S if it is continuous at each $(u_0^1, \dots, u_0^n)$ in S . The partial derivative $\partial \bar{u}^i / \partial u^j$ at $(u_0^1, \dots, u_0^n)$ is defined by the limit

$$\frac{\partial \bar{u}^i}{\partial u^j}(u_0^1, \dots, u_0^n) = \lim_{k \rightarrow 0} \frac{\bar{u}^i(u_0^1, \dots, u_0^j + k, \dots, u_0^n) - \bar{u}^i(u_0^1, \dots, u_0^n)}{k}$$

The underlying collection of points M together with the totality of allowable coordinate systems as defined above is called an *elementary coordinate manifold of n dimensions*.

Example 10.5.

- A regular simple (nonintersecting) curve in E^3 is an elementary coordinate manifold of dimension 1.
- An elementary (covered by a single patch) surface in E^3 is an elementary coordinate manifold of 2 dimensions.
- E^3 itself together with all coordinate systems $P(x^1, x^2, x^3)$ defined on E^3 for which the coordinate transformations $\bar{x}^i = \bar{x}^i(x^1, x^2, x^3)$, $i = 1, 2, 3$, and their inverses $x^i = x^i(\bar{x}^1, \bar{x}^2, \bar{x}^3)$, $i = 1, 2, 3$, are of class C^1 with Jacobians $\frac{\partial(x^1, x^2, x^3)}{\partial(\bar{x}^1, \bar{x}^2, \bar{x}^3)} \neq 0$ and $\frac{\partial(\bar{x}^1, \bar{x}^2, \bar{x}^3)}{\partial(x^1, x^2, x^3)} \neq 0$, is an elementary coordinate manifold of 3 dimensions.

TENSORS

A tensor T at a point P on a coordinate manifold can be thought of as a certain "object" attached to P with the following properties:

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With respect to each coordinate system $P(u^1, \dots, u^n)$ on the manifold, T is represented by a set of scalars C called the *components of T with respect to the coordinate system* $P(u^1, \dots, u^n)$.

(ii) If $P(\bar{u}^1, \dots, \bar{u}^n)$ is any other coordinate system on the manifold, the components $\bar{C}$ of T with respect to $P(\bar{u}^1, \dots, \bar{u}^n)$ are related to the components C by certain transformation laws which depend upon the coordinate transformation $\bar{u}^i = \bar{u}^i(u^1, \dots, u^n)$, $i = 1, \dots, n$, and its inverse $u^i = u^i(\bar{u}^1, \dots, \bar{u}^n)$.

An important example of a tensor at a point P on a surface is the one whose components with respect to a patch $\mathbf{x} = \mathbf{x}(u^1, u^2)$ containing P are the first fundamental coefficients with respect to P . If $\mathbf{x} = \mathbf{x}^*(\bar{u}^1, \bar{u}^2)$ is another patch containing P with first fundamental coefficients $\bar{g}_{ij}$, then it follows from equations (9.2) and (9.3) that the g_{ij} are related to the $\bar{g}_{ij}$ by the transformation law

$$\bar{g}_{ij} = g_{\alpha\beta} \frac{\partial u^\alpha}{\partial \bar{u}^i} \frac{\partial u^\beta}{\partial \bar{u}^j}, \quad \alpha, \beta = 1, 2 \quad (10.17)$$

This tensor is called the *covariant metric tensor* of the surface at P .

As for the metric tensor considered above, the tensor T will usually depend on the point P in the manifold, the components of $T(P)$ being given by functions of the coordinates $(u^1, \dots, u^n)$ of P . The function which assigns to each point P in the manifold the tensor $T(P)$ is called a *tensor field*.

Tensors are classified according to their transformation laws as follows:

(i) A tensor is called a *contravariant tensor of rank 1* or a *contravariant vector* if it has n components $A^1, \dots, A^n$ which transform in accordance with the law

$$\bar{A}^i = A^\alpha \frac{\partial \bar{u}^i}{\partial u^\alpha}, \quad \alpha, i = 1, \dots, n \quad (10.18)$$

(ii) A tensor is called a *covariant tensor of rank 1* or a *covariant vector* if it has n components $A_1, \dots, A_n$ which transform in accordance with the law

$$\bar{A}_i = A_\alpha \frac{\partial u^\alpha}{\partial \bar{u}^i}, \quad \alpha, i = 1, \dots, n \quad (10.19)$$

(iii) A tensor is called a *contravariant tensor of rank 2* if it has n^2 components A^{ij} , $i, j = 1, \dots, n$, which transform in accordance with the law

$$\bar{A}^{ij} = A^{\alpha\beta} \frac{\partial \bar{u}^i}{\partial u^\alpha} \frac{\partial \bar{u}^j}{\partial u^\beta}, \quad \alpha, \beta, i, j = 1, \dots, n \quad (10.20)$$

(iv) A tensor is called a *covariant tensor of rank 2* if it has n^2 components A_{ij} , $i, j = 1, \dots, n$, which transform in accordance with the law

$$\bar{A}_{ij} = A_{\alpha\beta} \frac{\partial u^\alpha}{\partial \bar{u}^i} \frac{\partial u^\beta}{\partial \bar{u}^j} \quad (10.21)$$

The above four tensors are special cases of

(v) A tensor is called a *mixed tensor, contravariant of order r and covariant of order s with weight N* , if it has n^{r+s} components $A^{i_1 \dots i_r}_{j_1 \dots j_s}$, $i_k, j_m = 1, \dots, n$, which transform in accordance with the law

$$\bar{A}^{i_1 \dots i_r}_{j_1 \dots j_s} = \left[\det \left(\frac{\partial u^i}{\partial \bar{u}^j} \right) \right]^N A^{\alpha_1 \dots \alpha_r}_{\beta_1 \dots \beta_s} \frac{\partial \bar{u}^{i_1}}{\partial u^{\alpha_1}} \dots \frac{\partial \bar{u}^{i_r}}{\partial u^{\alpha_r}} \frac{\partial u^{\beta_1}}{\partial \bar{u}^{j_1}} \dots \frac{\partial u^{\beta_s}}{\partial \bar{u}^{j_s}} \quad (10.22)$$

If the exponent N of the Jacobian $\det(\partial u^i/\partial \bar{u}^j)$ is zero, the tensor is called an *absolute* tensor. If $s = 0$, the tensor is purely contravariant. If $r = 0$, the tensor is purely covariant. The sum $r + s$ is called the *rank* of the tensor. Note that the tensors (i), (ii), (iii) and (iv) above are absolute tensors. We also define

(vi) A scalar is a tensor of rank zero.

Finally we note that the transformation law of a general tensor is *transitive*. For example, consider the transformation law $\bar{\bar{A}}^i = \bar{A}^\alpha \frac{\partial \bar{u}^i}{\partial \bar{u}^\alpha}$ which expresses the relation between the components of a contravariant vector in the $P(\bar{u}^1, \dots, \bar{u}^n)$ and $P(\bar{u}^1, \dots, \bar{u}^n)$ coordinate systems. From equation (10.18) it follows by substitution and the chain rule that

$$\bar{\bar{A}}^i = \left(A^\beta \frac{\partial \bar{u}^\alpha}{\partial u^\beta} \right) \frac{\partial \bar{u}^i}{\partial \bar{u}^\alpha} = A^\beta \left(\frac{\partial \bar{u}^i}{\partial \bar{u}^\alpha} \frac{\partial \bar{u}^\alpha}{\partial u^\beta} \right) = A^\beta \frac{\partial \bar{u}^i}{\partial u^\beta}$$

which is exactly the transformation law relating the components of the vector with respect to the $P(\bar{u}^1, \dots, \bar{u}^n)$ and $P(u^1, \dots, u^n)$ coordinate systems.

Example 10.6.

(a) Let $f(x^1, x^2, x^3)$ be a real valued function of class C^1 defined on an open set U in E^3 . If $x^i = x^i(\bar{x}^1, \bar{x}^2, \bar{x}^3)$, $i = 1, 2, 3$, is an allowable coordinate transformation, then it follows from the chain rule that at each point

$$\frac{\partial f}{\partial \bar{x}^i} = \frac{\partial f}{\partial x^\alpha} \frac{\partial x^\alpha}{\partial \bar{x}^i}, \quad i = 1, 2, 3$$

Thus from equation (10.19) we see that the $\partial f/\partial x^i$, $i = 1, 2, 3$, are the components of a covariant vector field on U called the *gradient* of f .

(b) It follows from equations (10.17) and (10.21) that the covariant metric tensor, i.e. the tensor field on a surface whose components are the first fundamental coefficients g_{ik} , is an absolute covariant tensor field of rank 2.

(c) We recall (see equation (9.11), page 175) that the second fundamental coefficients b_{ij} transform in the same way as the g_{ij} , i.e. on the intersection of two patches $\mathbf{x} = \mathbf{x}(u^1, u^2)$ and $\mathbf{x} = \mathbf{x}^*(\bar{u}^1, \bar{u}^2)$ we have

$$\bar{b}_{ij} = b_{\alpha\beta} \frac{\partial u^\alpha}{\partial \bar{u}^i} \frac{\partial u^\beta}{\partial \bar{u}^j}$$

Thus the second fundamental coefficients are also the components of an absolute covariant tensor field on the surface of rank 2.

(d) Let $\bar{u}^i = \bar{u}^i(u^1, \dots, u^n)$ be an allowable coordinate transformation on a coordinate manifold of dimension n and consider at a point the sum $\delta^\beta_\alpha \frac{\partial \bar{u}^j}{\partial u^\beta} \frac{\partial u^\alpha}{\partial \bar{u}^i}$. Since the only contribution occurs when $\alpha = \beta$, we have

$$\delta^\beta_\alpha \frac{\partial \bar{u}^j}{\partial u^\beta} \frac{\partial u^\alpha}{\partial \bar{u}^i} = \frac{\partial \bar{u}^j}{\partial u^\alpha} \frac{\partial u^\alpha}{\partial \bar{u}^i} = \delta^j_i = \bar{\delta}^j_i$$

where we used equation (10.16). It follows from equation (10.22) that the Kronecker delta δ^j_i is a mixed absolute tensor of rank 2, covariant of order 1 and contravariant of order 1. Note that this is a tensor whose components are the same with respect to every coordinate system on the manifold.

(e) At a point on the intersection of two patches on a surface let $A^{ij} = g^{\alpha\beta} \frac{\partial \bar{u}^i}{\partial u^\alpha} \frac{\partial \bar{u}^j}{\partial u^\beta}$ where the g^{ij} are defined in equation (10.11), and consider the sum

$$\begin{aligned} \bar{g}_{ik} A^{kj} &= g_{\gamma\sigma} \frac{\partial u^\gamma}{\partial \bar{u}^i} \frac{\partial u^\sigma}{\partial \bar{u}^k} g^{\alpha\beta} \frac{\partial \bar{u}^k}{\partial u^\alpha} \frac{\partial \bar{u}^j}{\partial u^\beta} = \left(\frac{\partial u^\sigma}{\partial \bar{u}^k} \frac{\partial \bar{u}^k}{\partial u^\alpha} \right) g_{\gamma\sigma} g^{\alpha\beta} \frac{\partial u^\gamma}{\partial \bar{u}^i} \frac{\partial \bar{u}^j}{\partial u^\beta} \\ &= \delta^\sigma_\alpha g_{\gamma\sigma} g^{\alpha\beta} \frac{\partial u^\gamma}{\partial \bar{u}^i} \frac{\partial \bar{u}^j}{\partial u^\beta} = g_{\gamma\alpha} g^{\alpha\beta} \frac{\partial u^\gamma}{\partial \bar{u}^i} \frac{\partial \bar{u}^j}{\partial u^\beta} = \delta^\beta_\gamma \frac{\partial u^\gamma}{\partial \bar{u}^i} \frac{\partial \bar{u}^j}{\partial u^\beta} \\ &= \frac{\partial u^\beta}{\partial \bar{u}^i} \frac{\partial \bar{u}^j}{\partial u^\beta} = \delta^j_i = \bar{\delta}^j_i \end{aligned}$$

Since $\bar{g}_{ik}A^{kj} = \delta_i^j$ it follows that

$$\bar{g}^{ij} = A^{ij} = g^{\alpha\beta} \frac{\partial \bar{u}^i}{\partial u^\alpha} \frac{\partial \bar{u}^j}{\partial u^\beta}$$

Thus from equation (10.20) we see that the g^{ij} are the components of an absolute contravariant tensor field on the surface of rank 2, called the *contravariant metric tensor*.

Consider the 27 scalars e^{ijk} , $i, j, k = 1, 2, 3$, defined as follows:

$$e^{ijk} = \begin{cases} 0 & \text{if two indices } i, j, k \text{ are the same} \\ 1 & \text{if } i, j, k \text{ is an even permutation of } 1, 2, 3 \\ -1 & \text{if } i, j, k \text{ is an odd permutation of } 1, 2, 3 \end{cases}$$

For example $e^{112} = 0$, $e^{212} = 0$, $e^{123} = 1$, $e^{213} = -1$, $e^{231} = 1$. We recall that

$$\det(\alpha_i^j) = \det \begin{pmatrix} \alpha_1^1 & \alpha_1^2 & \alpha_1^3 \\ \alpha_2^1 & \alpha_2^2 & \alpha_2^3 \\ \alpha_3^1 & \alpha_3^2 & \alpha_3^3 \end{pmatrix} = \sum \pm \alpha_i^1 \alpha_j^2 \alpha_k^3$$

where + is taken when i, j, k is an even permutation of 1, 2, 3 and - is taken when i, j, k is an odd permutation of 1, 2, 3. But then it follows from the definition of e^{ijk} that we can write

$$\det(\alpha_i^j) = \sum_{\alpha\beta\gamma} e^{\alpha\beta\gamma} \alpha_\alpha^1 \alpha_\beta^2 \alpha_\gamma^3 = e^{\alpha\beta\gamma} \alpha_\alpha^1 \alpha_\beta^2 \alpha_\gamma^3$$

where we sum over all α, β, γ . We note further that

$$e^{\alpha\beta\gamma} \alpha_\alpha^p \alpha_\beta^q \alpha_\gamma^r = \begin{cases} +\det(\alpha_i^j) & \text{if } p, q, r \text{ is an even permutation of } 1, 2, 3 \\ -\det(\alpha_i^j) & \text{if } p, q, r \text{ is an odd permutation of } 1, 2, 3 \end{cases}$$

$$e^{pqr} \det(\alpha_i^j) = e^{\alpha\beta\gamma} \alpha_\alpha^p \alpha_\beta^q \alpha_\gamma^r$$

Thus

Now let $\bar{u}^i = \bar{u}^i(u^1, u^2, u^3)$ be a coordinate transformation on a coordinate manifold of 3 dimensions.

Then it follows from the above that

$$e^{pqr} \det \left(\frac{\partial \bar{u}^i}{\partial u^i} \right) = e^{\alpha\beta\gamma} \frac{\partial \bar{u}^p}{\partial u^\alpha} \frac{\partial \bar{u}^q}{\partial u^\beta} \frac{\partial \bar{u}^r}{\partial u^\gamma}$$

or $\bar{e}^{pqr} = e^{pqr} = \left[\det \left(\frac{\partial \bar{u}^i}{\partial u^i} \right) \right]^{-1} e^{\alpha\beta\gamma} \frac{\partial \bar{u}^p}{\partial u^\alpha} \frac{\partial \bar{u}^q}{\partial u^\beta} \frac{\partial \bar{u}^r}{\partial u^\gamma} = \det \left(\frac{\partial \bar{u}^j}{\partial \bar{u}^i} \right) e^{\alpha\beta\gamma} \frac{\partial \bar{u}^p}{\partial u^\alpha} \frac{\partial \bar{u}^q}{\partial u^\beta} \frac{\partial \bar{u}^r}{\partial u^\gamma}$

It follows that the e^{ijk} are the components of a contravariant tensor of rank 3 and weight 1. We note that the quantities e_{ijk} , $i, j, k = 1, 2, 3$, where $e_{ijk} = e^{ijk}$, are the components of a covariant tensor of rank 3 and weight -1. The proof of this is left to the reader as an exercise.

A set of components of a tensor is said to be *symmetric* with respect to two contravariant indices (superscripts) or with respect to two covariant indices (subscripts) if the components remain the same when the indices are interchanged. For example A_{pq}^{ijk} is symmetric with respect to the first and third contravariant indices if $A_{pq}^{ijk} = A_{pq}^{kji}$ for all i and k .

The component of a tensor is said to be *skew-symmetric* with respect to two contravariant indices or with respect to two covariant indices if the algebraic signs of the components change when the first and third contravariant indices with respect to all pairs of variant indices or with respect to two covariant indices are interchanged. Thus A_{pq}^{ijk} is skew-symmetric with respect to the first and third contravariant indices if $A_{pq}^{ijk} = -A_{pq}^{kji}$ for all i, k .

A tensor is said simply to be symmetric if it is symmetric with respect to all pairs of contravariant and all pairs of covariant indices. If the tensor is skew-symmetric with respect to all pairs of contravariant and all pairs of covariant indices it is said to be skew-symmetric.

In Problem 10.19 we show that if the components of a tensor with respect to one coordinate system are symmetric with respect to a pair of indices, then the components of the tensor are symmetric with respect to the same indices in any other coordinate system. Thus the property of being symmetric with respect to a pair of indices is a property of the tensor. The same is true for skew-symmetry.

Example 10.7.

(a) Both the covariant and contravariant metric tensors are symmetric tensors, since $g^{ij} = g^{ji}$, $i, j = 1, 2$, and $g_{ij} = g_{ji}$, $i, j = 1, 2$.

(b) The Kronecker delta can be extended as follows:

$$\delta_{ij}^{pq} = \begin{cases} 1 & \text{if } i = p, j = q, \text{ and } i \neq j \\ -1 & \text{if } i = q, j = p, \text{ and } i \neq j \\ 0 & \text{otherwise} \end{cases} \quad (i, j, p, q = 1, \dots, n)$$

In Problem 10.14, page 220, we show that the δ_{ij}^{pq} are the components of an absolute tensor of rank 4, covariant of order 2 and contravariant of order 2. It is clearly skew-symmetric with respect to its contravariant indices. For if $i = p, j = q$, and $i \neq j$, $\delta_{ij}^{pq} = 1$ and $\delta_{ij}^{pq} = -1$; hence $\delta_{ij}^{pq} = -\delta_{ij}^{qp}$. If $i = q, j = p$ and $i \neq j$, then $\delta_{ij}^{pq} = -1$, $\delta_{ij}^{qp} = 1$ and $\delta_{ij}^{pq} = -\delta_{ij}^{qp}$. Otherwise $\delta_{ij}^{pq} = 0$, $\delta_{ij}^{qp} = 0$ and $\delta_{ij}^{pq} = -\delta_{ij}^{qp}$. Similarly it is skew-symmetric with respect to its covariant indices.

(c) The above can be generalized as follows:

$$\delta_{i_1 \dots i_m}^{j_1 \dots j_m} = \begin{cases} 1 & \text{if the } i_1, \dots, i_m \text{ are distinct and } j_1, \dots, j_m \text{ is an even permutation of } i_1, \dots, i_m \\ -1 & \text{if the } i_1, \dots, i_m \text{ are distinct and } j_1, \dots, j_m \text{ is an odd permutation of } i_1, \dots, i_m \\ 0 & \text{otherwise} \end{cases}$$

For example,

$$\delta_{1122}^{1245} = 0, \quad \delta_{1234}^{1134} = 0, \quad \delta_{1234}^{5678} = 0, \quad \delta_{1234}^{2134} = -1, \quad \delta_{1234}^{2314} = +1, \quad \delta_{3456}^{3465} = -1, \quad \delta_{4356}^{4356} = 1$$

It can be shown that the $\delta_{i_1 \dots i_m}^{j_1 \dots j_m}$ are the components of a skew-symmetric absolute tensor covariant of order m and contravariant of order m .

TENSOR ALGEBRA

(a) *Addition.* Let $A_{j_1 \dots j_s}^{i_1 \dots i_r}$ and $B_{j_1 \dots j_s}^{i_1 \dots i_r}$ be the components of two tensors A and B contravariant and covariant of the same order and of the same weight. In Problem 10.18, page 221, we show that the sum

$$C_{j_1 \dots j_s}^{i_1 \dots i_r} = A_{j_1 \dots j_s}^{i_1 \dots i_r} + B_{j_1 \dots j_s}^{i_1 \dots i_r}$$

obtained by adding corresponding components of A and B are the components of a tensor C contravariant and covariant of the same orders and of the same weight as A and B . The tensor C is called the *sum* of A and B .

(b) *Outer product of tensors.* If the components $B_{\beta_1 \dots \beta_q}^{\alpha_1 \dots \alpha_p}$ of a tensor B of rank $p + q$ are multiplied by the component $A_{j_1 \dots j_s}^{i_1 \dots i_r}$ of a tensor A of rank $r + s$, the result will be a set of $n^{r+s+p+q}$ numbers

$$C_{j_1 \dots j_s \beta_1 \dots \beta_q}^{i_1 \dots i_r \alpha_1 \dots \alpha_p} = A_{j_1 \dots j_s}^{i_1 \dots i_r} B_{\beta_1 \dots \beta_q}^{\alpha_1 \dots \alpha_p}$$

It is easily verified that the $C_{j_1 \dots j_s \beta_1 \dots \beta_q}^{i_1 \dots i_r \alpha_1 \dots \alpha_p}$ are the components of a tensor C , contravariant of order $r + p$, covariant of order $s + q$, and of weight $N_1 + N_2$, where N_1 is the weight of A and N_2 is the weight of B . The tensor C is called the *outer product* of A and B . As a special case A can be a tensor of rank 0, i.e. $A = \text{scalar}$.

(c) *Contraction.* Let $A_{j_1, \dots, j_r}^{i_1, \dots, i_r}$ be the components of a tensor A contravariant of order r , covariant of order s , and of weight N . The set of n^{r+s-2} scalars

$$B_{j_1, \dots, j_s}^{i_1, \dots, i_r} = A_{a_1, \dots, a_s}^{a_1, \dots, a_r, i_1, \dots, i_r}$$

obtained by identifying the first contravariant index with the first covariant index and summing, can be shown to be the components of a tensor B , contravariant of order $r-1$, covariant of order $s-1$ and of weight N . The tensor B is said to be a contraction of the tensor A . Such a contracted tensor can be formed for any choice of one covariant and one contravariant index.

Example 10.8.

(a) Let A^{ij} be the components of an arbitrary contravariant tensor of rank 2, and let $B^{ij} = A^{ij}$. Note that the B^{ij} are also the components of a contravariant tensor of rank 2, since

$$B^{\alpha\beta} \det \left(\frac{\partial u^i}{\partial \tilde{u}^j} \right)^N \frac{\partial \tilde{u}^i}{\partial u^\alpha} \frac{\partial \tilde{u}^j}{\partial u^\beta} = A^{\beta\alpha} \det \left(\frac{\partial u^i}{\partial \tilde{u}^j} \right)^N \frac{\partial \tilde{u}^i}{\partial u^\alpha} \frac{\partial \tilde{u}^j}{\partial u^\beta} = A^{\beta\alpha} \det \left(\frac{\partial u^i}{\partial \tilde{u}^j} \right)^N \frac{\partial \tilde{u}^j}{\partial u^\beta} \frac{\partial \tilde{u}^i}{\partial u^\alpha} = \tilde{A}^{\beta\alpha} = \tilde{B}^{\beta\alpha}$$

Now consider the tensors defined by

$$C^{ij} = \frac{1}{2}(A^{ij} + B^{ij})$$

$$D^{ij} = \frac{1}{2}(A^{ij} - B^{ij})$$

$$C^{ji} = \frac{1}{2}(A^{ji} + B^{ji}) = \frac{1}{2}(B^{ij} + A^{ij}) = C^{ij}$$

Here

$$D^{ji} = \frac{1}{2}(A^{ji} - B^{ji}) = \frac{1}{2}(B^{ij} - A^{ij}) = -D^{ij}$$

and

Thus C^{ij} is symmetric and D^{ij} is skew-symmetric. But $C^{ij} + D^{ij} = A^{ij}$ and so every contravariant tensor of rank 2 is the sum of a symmetric and a skew-symmetric contravariant tensor of rank 2.

Let $e^{11} = e_{11} = 0$, $e^{12} = e_{12} = 1$, $e^{21} = e_{21} = -1$ and $e^{22} = e_{22} = 0$. Here e^{ij} and e_{ij} are the two dimensional analogs of e^{ijk} and e_{ijk} in Example 10.6(f). Now consider the outer product

$$A_{pq}^{ij} = e^{ij} e_{pq}$$

(b) Let $e^{11} = e_{11} = 0$, $e^{12} = e_{12} = 1$, $e^{21} = e_{21} = -1$ and $e^{22} = e_{22} = 0$. Here e^{ij} and e_{ij} are the two dimensional analogs of e^{ijk} and e_{ijk} in Example 10.6(f). Now consider the outer product

Observe that if the $p; q$ are distinct, then $A_{pq}^{ij} = 1$ if $i = p$ and $j = q$, and $A_{pq}^{ij} = -1$ if $i = q$ and $j = p$. Otherwise $A_{pq}^{ij} = 0$. Namely from Example 10.7(b),

$$A_{pq}^{ij} = e^{ij} e_{pq} = \delta_{pq}^{ij}$$

We see further that the contraction

$$\delta_{\alpha\alpha}^{c\dot{c}} = e^{c\dot{c}} e_{c\alpha} = e^{1j} e_{1\alpha} + e^{2j} e_{2\alpha} = \begin{cases} 1 & \text{if } j = \alpha \\ 0 & \text{otherwise} \end{cases} = \delta_{\alpha}^j$$

APPLICATIONS OF TENSORS TO THE EQUATIONS OF SURFACE THEORY

(10.23)

We consider the Gauss equations

$$\mathbf{x}_{ij} = \Gamma_{ij}^{\alpha} \mathbf{x}_{\alpha} + b_{ij} \mathbf{N} \quad (\alpha, i, j = 1, 2)$$

We consider the Gauss equations with $\mathbf{x}_k$, we obtain

$$\mathbf{x}_{ij} \cdot \mathbf{x}_k = \Gamma_{ij}^{\alpha} (\mathbf{x}_{\alpha} \cdot \mathbf{x}_k) = \Gamma_{ij}^{\alpha} g_{\alpha k}$$

By taking the scalar product of the Gauss equations with $\mathbf{x}_k$, we obtain the *Christoffel symbols of the first kind*. Using

The quantities $\Gamma_{ijk} \equiv (\mathbf{x}_{ij} \cdot \mathbf{x}_k)$ are called the

$g_{\alpha\beta} g^{\beta k} = \delta_{\alpha}^k$, it follows further that $\Gamma_{ij\beta} g^{\beta k} = \Gamma_{ij}^{\alpha} g_{\alpha\beta} g^{\beta k} = \Gamma_{ij}^{\alpha} \delta_{\alpha}^k = \Gamma_{ij}^k$ are related to the Christoffel symbols of the

(10.24)

Thus the Christoffel symbols of the first kind are related to the Christoffel symbols of the second kind by the equations $\Gamma_{ijk} = g_{k\alpha} \Gamma_{ij}^{\alpha}$ and $\Gamma_{ij}^k = g^{k\alpha} \Gamma_{i\alpha}^j$

In Problem 10.24, page 223, we prove that the Γ_{ijk} are given by

$$\Gamma_{ijk} = \frac{1}{2} \left[\frac{\partial g_{jk}}{\partial u^i} + \frac{\partial g_{ki}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^k} \right] \quad (10.25)$$

and hence the Γ_{ij}^k are given by

$$\Gamma_{ij}^k = \frac{1}{2} g^{k\alpha} \left[\frac{\partial g_{j\alpha}}{\partial u^i} + \frac{\partial g_{\alpha i}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^\alpha} \right] \quad (10.26)$$

We note that the Christoffel symbols are *not* the components of a tensor as previously defined. Their transformation laws involve the second derivatives of the parameter transformation. In Problem 10.27, page 223, we prove

Theorem 10.8. The Christoffel symbols transform in accordance with the laws

$$\begin{aligned} \bar{\Gamma}_{ij}^k &= \left[\Gamma_{\alpha\beta}^\gamma \frac{\partial u^\alpha}{\partial \bar{u}^i} \frac{\partial u^\beta}{\partial \bar{u}^j} + \frac{\partial^2 u^\gamma}{\partial \bar{u}^i \partial \bar{u}^j} \right] \frac{\partial \bar{u}^k}{\partial u^\gamma} \\ \bar{\Gamma}_{ijk} &= \left[\Gamma_{\alpha\beta\gamma} \frac{\partial u^\alpha}{\partial \bar{u}^i} \frac{\partial u^\beta}{\partial \bar{u}^j} + g_{\alpha\gamma} \frac{\partial^2 u^\alpha}{\partial \bar{u}^i \partial \bar{u}^j} \right] \frac{\partial u^\gamma}{\partial \bar{u}^k} \end{aligned}$$

We consider next the Weingarten equations

$$N_i = \beta_i^\alpha \mathbf{x}_\alpha, \quad \alpha, i = 1, 2 \quad (10.27)$$

By taking the scalar product with $\mathbf{x}_j$, we find that the second fundamental coefficients b_{ij} satisfy

$$-b_{ij} = N_i \cdot \mathbf{x}_j = \beta_i^\alpha \mathbf{x}_\alpha \cdot \mathbf{x}_j = \beta_i^\alpha g_{\alpha j}$$

If we define $b_i^j = b_{i\alpha} g^{\alpha j}$, it follows that

$$b_i^j = b_{i\gamma} g^{\gamma j} = -\beta_i^\alpha g_{\alpha\gamma} g^{\gamma j} = -\beta_i^\alpha \delta_\alpha^j = -\beta_i^j$$

Thus the Weingarten equations can be written as

$$N_i = -b_i^\alpha \mathbf{x}_\alpha, \quad i = 1, 2 \quad (10.28)$$

where the b_i^j and b_{ij} are related by

$$b_i^j = g^{\alpha j} b_{i\alpha} \quad \text{and} \quad b_{ij} = g_{\alpha j} b_i^\alpha \quad (10.29)$$

Here the b_{ij} are the components of an absolute covariant tensor of rank 2 and the b_i^j are the components of an absolute tensor of rank 2 contravariant of order 1 and covariant of order 1.

We now define the *Riemann symbols of the second kind*,

$$R_{mijk} = b_{ik} b_{jm} - b_{ij} b_{km} \quad (10.30)$$

and the associated *Riemann symbols of the first kind*,

$$R_{ijk}^p = g^{\alpha p} R_{\alpha ijk} \quad (10.31)$$

We see that the R_{mijk} are the components of an absolute covariant tensor of rank 4 and that the R_{ijk}^p are the components of an absolute tensor of rank 4 covariant of order 3 and contravariant of order 1, called the *covariant Riemann curvature tensor* and *mixed Riemann curvature tensor* respectively. From equations (10.29) and (10.31) it follows that

$$R_{ijk}^p = g^{\alpha p} (b_{ik} b_{j\alpha} - b_{ij} b_{k\alpha}) = b_{ik} b_j^p - b_{ij} b_k^p \quad (10.32)$$

Observe in (10.30) that the R_{mijk} are skew-symmetric with respect to the first two indices and the last two indices, i.e.,

$$R_{imjk} = -R_{mijk} \quad \text{and} \quad R_{mijk} = -R_{mjik} \quad (10.33)$$

$R_{mijk} = 0$ whenever the first two or last two indices are the same. Thus only four components are different from zero. They are

$$R_{1212} = R_{2121} = b_{22}b_{11} - b_{12}b_{21} = LN - M^2 = b \quad (10.34)$$

$$R_{1221} = R_{2112} = b_{12}b_{21} - b_{22}b_{11} = -(LN - M^2) = -b \quad (10.35)$$

Although the curvature tensors were defined in terms of the coefficients of the second fundamental form, they can in fact be expressed only in terms of the coefficients of the fundamental form, i.e. the metric tensors, and their derivatives. In Problem 10.29, § 224, we prove

$$R_{mijk} = \frac{\partial}{\partial u^j} \Gamma_{ikm} - \frac{\partial}{\partial u^k} \Gamma_{ijm} + \Gamma_{ij}^{\alpha} \Gamma_{mk\alpha} - \Gamma_{ik}^{\alpha} \Gamma_{mj\alpha}$$

Problem 10.9.

Since the Christoffel symbols depend only upon the metric tensors and their derivatives, the same is true for the curvature tensors. We note that this is equivalent follows that the same is true for the curvature tensors. We note that this is equivalent Gauss's theorem, since from equation (10.34) the Gaussian curvature

$$K = \frac{LN - M^2}{EG - F^2} = \frac{b}{g} = \frac{R_{1212}}{g}$$

Solved Problems

THEORY OF SURFACES

1. Show that the Gauss-Weingarten equations for a Monge patch $\mathbf{x} = ue_1 + ve_2 + f(u, v)e_3$ are

$$g_{xu} = prx_u + qrx_v + rg^{1/2}N$$

$$g_{xv} = psx_u + qsx_v + sg^{1/2}N$$

$$g_{xv} = ptx_u + qtx_v + tg^{1/2}N$$

$$g_{vv} = prx_u + qrx_v + rg^{1/2}N$$

$$\text{where } p = f_u, q = f_v, r = f_{uu}, s = f_{uv}, t = f_{vv}, g = 1 + p^2 + q^2.$$

$$x_u = e_1 + pe_3, x_v = e_2 + qe_3, x_{uu} = re_3, x_{uv} = se_3, x_{vv} = te_3$$

$$E = x_u \cdot x_u = 1 + p^2, F = x_u \cdot x_v = pq, G = x_v \cdot x_v = 1 + q^2$$

$$EG - F^2 = 1 + p^2 + q^2 = g, N = x_u \times x_v / |x_u \times x_v| = -(pe_1 + qe_2 - e_3)/g^{1/2}$$

$$L = x_{uu} \cdot N = r/g^{1/2}, M = x_{uv} \cdot N = s/g^{1/2}, N = x_{vv} \cdot N = t/g^{1/2}$$

$$E_u = 2pr, E_v = 2ps, F_u = ps + qr, F_v = pt + qs, G_u = 2qs, G_v = 2qt$$

$$E_u = 2pr, E_v = 2ps, F_u = ps + qr, F_v = pt + qs, G_u = 2qs, G_v = 2qt$$

$$\Gamma_{11}^1 = pr/g, \Gamma_{12}^1 = ps/g, \Gamma_{22}^1 = pt/g$$

$$\Gamma_{11}^2 = qr/g, \Gamma_{12}^2 = qs/g, \Gamma_{22}^2 = qt/g$$

$$\beta_1^1 = (spq - rq^2 - r)/g^{3/2}, \beta_2^1 = (tpq - sq^2 - s)/g^{3/2}$$

$$\beta_1^2 = (spq - rp^2 - p)/g^{3/2}, \beta_2^2 = (spq - tp^2 - t)/g^{3/2}$$

$$\beta_1^1 = (spq - rq^2 - r)/g^{3/2}, \beta_2^1 = (tpq - sq^2 - s)/g^{3/2}$$

$$\beta_1^2 = (spq - rp^2 - p)/g^{3/2}, \beta_2^2 = (spq - tp^2 - t)/g^{3/2}$$

from which the result follows.

* 10.2. Using the Weingarten equations, show that

$$\text{III} - 2\text{HII} + \text{KI} = 0$$

where the third fundamental form $\text{III} = d\mathbf{N} \cdot d\mathbf{N}$, H is the mean curvature and K is the Gaussian curvature.

Using equation (10.2), page 201, we obtain

$$\begin{aligned} \mathbf{N}_u \cdot \mathbf{N}_u &= (\beta_1^1 \mathbf{x}_u + \beta_1^2 \mathbf{x}_v) \cdot (\beta_1^1 \mathbf{x}_u + \beta_1^2 \mathbf{x}_v) \\ &= \frac{(MF - LG)^2 E}{(EG - F^2)^2} + \frac{2(MF - LG)(LF - ME)F}{(EG - F^2)^2} + \frac{(LF - ME)^2 G}{(EG - F^2)^2} \\ &= \frac{(-2LMF + L^2G + EM^2)(EG - F^2)}{(EG - F^2)^2} = \frac{(EN - 2MF + LG)L - (LN - M^2)E}{EG - F^2} \\ &= 2HL - KE \end{aligned}$$

Similarly

$$\begin{aligned} \mathbf{N}_u \cdot \mathbf{N}_v &= (\beta_1^1 \mathbf{x}_u + \beta_1^2 \mathbf{x}_v) \cdot (\beta_2^1 \mathbf{x}_u + \beta_2^2 \mathbf{x}_v) \\ &= \frac{(MF - LG)(NF - MG)E}{(EG - F^2)^2} + \frac{(NF - MG)(LF - ME)F}{(EG - F^2)^2} \\ &\quad + \frac{(MF - LG)(MF - NE)F}{(EG - F^2)^2} + \frac{(LF - ME)(MF - NE)G}{(EG - F^2)^2} \\ &= \frac{(MEN - M^2F + LGM - FLN)(EG - F^2)}{(EG - F^2)^2} = \frac{(EN - 2MF + LG)M}{EG - F^2} - \frac{(LN - M^2)F}{EG - F^2} \\ &= 2HM - KF \end{aligned}$$

$$\begin{aligned} \text{Also } \mathbf{N}_v \cdot \mathbf{N}_v &= (\beta_2^1 \mathbf{x}_u + \beta_2^2 \mathbf{x}_v) \cdot (\beta_2^1 \mathbf{x}_u + \beta_2^2 \mathbf{x}_v) \\ &= \frac{(NF - MG)^2 E}{(EG - F^2)^2} + \frac{2(NF - MG)(MF - NE)F}{(EG - F^2)^2} + \frac{(MF - NE)^2 G}{(EG - F^2)^2} \\ &= \frac{(EN^2 - 2MFN + M^2G)(EG - F^2)}{(EG - F^2)^2} = \frac{(EN - 2MF + LG)N}{EG - F^2} - \frac{(LN - M^2)G}{EG - F^2} \\ &= 2HN - KG \end{aligned}$$

It follows that

$$\begin{aligned} \text{III} &= d\mathbf{N} \cdot d\mathbf{N} = (\mathbf{N}_u du + \mathbf{N}_v dv) \cdot (\mathbf{N}_u du + \mathbf{N}_v dv) = \mathbf{N}_u \cdot \mathbf{N}_u du^2 + 2\mathbf{N}_u \cdot \mathbf{N}_v du dv + \mathbf{N}_v \cdot \mathbf{N}_v dv^2 \\ &= (2HL - KE) du^2 + 2(2HM - KF) du dv + (2HN - KG) dv^2 \\ &= 2H(L du^2 + 2M du dv + N dv^2) - K(E du^2 + 2F du dv + G dv^2) = 2\text{HII} - \text{KI} \end{aligned}$$

which gives the required result.

10.3. Prove that the Christoffel symbols Γ_{ij}^k are given by equations (10.4), page 202.

Observe that

$$\begin{aligned} \mathbf{x}_u \cdot \mathbf{x}_{uu} &= \frac{1}{2}(\mathbf{x}_u \cdot \mathbf{x}_u)_u = \frac{1}{2}E_u, & \mathbf{x}_u \cdot \mathbf{x}_{uv} &= \frac{1}{2}(\mathbf{x}_u \cdot \mathbf{x}_v)_v = \frac{1}{2}E_v \\ \mathbf{x}_v \cdot \mathbf{x}_{vv} &= \frac{1}{2}(\mathbf{x}_v \cdot \mathbf{x}_v)_v = \frac{1}{2}G_v, & \mathbf{x}_v \cdot \mathbf{x}_{uv} &= \frac{1}{2}(\mathbf{x}_v \cdot \mathbf{x}_u)_u = \frac{1}{2}G_u \end{aligned}$$

Also using the above,

$$\begin{aligned} F_u &= (\mathbf{x}_u \cdot \mathbf{x}_v)_u = \mathbf{x}_{uu} \cdot \mathbf{x}_v + \mathbf{x}_u \cdot \mathbf{x}_{uv} = \mathbf{x}_{uu} \cdot \mathbf{x}_v + \frac{1}{2}E_v \\ F_v &= (\mathbf{x}_u \cdot \mathbf{x}_v)_v = \mathbf{x}_{uv} \cdot \mathbf{x}_v + \mathbf{x}_u \cdot \mathbf{x}_{vv} = \frac{1}{2}G_u + \mathbf{x}_u \cdot \mathbf{x}_{vv} \end{aligned}$$

Hence

$$\mathbf{x}_v \cdot \mathbf{x}_{uu} = F_u - \frac{1}{2}E_v, \quad \mathbf{x}_u \cdot \mathbf{x}_{vv} = F_v - \frac{1}{2}G_u$$

Now from the Gauss equations and the above,

$$\begin{aligned} \frac{1}{2}E_u &= \mathbf{x}_u \cdot \mathbf{x}_{uu} = \Gamma_{11}^1 \mathbf{x}_u \cdot \mathbf{x}_u + \Gamma_{11}^2 \mathbf{x}_u \cdot \mathbf{x}_v = \Gamma_{11}^1 E + \Gamma_{11}^2 F \\ F_u - \frac{1}{2}E_v &= \mathbf{x}_v \cdot \mathbf{x}_{uu} = \Gamma_{11}^1 \mathbf{x}_v \cdot \mathbf{x}_u + \Gamma_{11}^2 \mathbf{x}_v \cdot \mathbf{x}_v = \Gamma_{11}^1 F + \Gamma_{11}^2 G \\ \frac{1}{2}E_v &= \mathbf{x}_u \cdot \mathbf{x}_{uv} = \Gamma_{12}^1 \mathbf{x}_u \cdot \mathbf{x}_u + \Gamma_{12}^2 \mathbf{x}_u \cdot \mathbf{x}_v = \Gamma_{12}^1 E + \Gamma_{12}^2 F \\ \frac{1}{2}G_u &= \mathbf{x}_v \cdot \mathbf{x}_{uv} = \Gamma_{12}^1 \mathbf{x}_v \cdot \mathbf{x}_u + \Gamma_{12}^2 \mathbf{x}_v \cdot \mathbf{x}_v = \Gamma_{12}^1 F + \Gamma_{12}^2 G \\ F_v - \frac{1}{2}G_u &= \mathbf{x}_u \cdot \mathbf{x}_{vv} = \Gamma_{22}^1 \mathbf{x}_u \cdot \mathbf{x}_u + \Gamma_{22}^2 \mathbf{x}_u \cdot \mathbf{x}_v = \Gamma_{22}^1 E + \Gamma_{22}^2 F \\ \frac{1}{2}G_v &= \mathbf{x}_v \cdot \mathbf{x}_{vv} = \Gamma_{22}^1 \mathbf{x}_v \cdot \mathbf{x}_u + \Gamma_{22}^2 \mathbf{x}_v \cdot \mathbf{x}_v = \Gamma_{22}^1 F + \Gamma_{22}^2 G \end{aligned}$$

Solving the first two equations for Γ_{11}^1 and Γ_{11}^2 , the second two equations for Γ_{12}^1 and Γ_{12}^2 , and the last two equations for Γ_{22}^1 and Γ_{22}^2 , we obtain

$$\Gamma_{11}^1 = \frac{GE_u - 2FF_u + FE_v}{2(EG - F^2)}$$

$$\Gamma_{12}^1 = \frac{GE_v - FG_u}{2(EG - F^2)}$$

$$\Gamma_{22}^1 = \frac{2GF_v - GG_u - FG_v}{2(EG - F^2)}$$

$$\Gamma_{11}^2 = \frac{2EF_u - EE_v + FE_u}{2(EG - F^2)}$$

$$\Gamma_{12}^2 = \frac{EG_u - FE_v}{2(EG - F^2)}$$

$$\Gamma_{22}^2 = \frac{EG_v - 2FF_v + FG_u}{2(EG - F^2)}$$

which are the required results.

104. Prove that $K(EG - F^2)^2 = [x_{uu}x_u x_v][x_{vv}x_u x_v] - [x_{uv}x_u x_v]^2$.

$$L = x_{uu} \cdot N = x_{uu} \cdot x_u \times x_v / |x_u \times x_v| = [x_{uu}x_u x_v] / |x_u \times x_v|$$

$$M = x_{uv} \cdot N = [x_{uv}x_u x_v] / |x_u \times x_v|, \quad N = x_{vv} \cdot N = [x_{vv}x_u x_v] / |x_u \times x_v|$$

$$LN - M^2 = \frac{[x_{uu}x_u x_v][x_{vv}x_u x_v] - [x_{uv}x_u x_v]^2}{|x_u \times x_v|^2}$$

Thus

$$|x_u \times x_v|^2 = (x_u \times x_v) \cdot (x_u \times x_v) = (x_u \cdot x_u)(x_v \cdot x_v) - (x_u \cdot x_v)^2 = EG - F^2$$

Also

$$K = \frac{LN - M^2}{EG - F^2} = \frac{[x_{uu}x_u x_v][x_{vv}x_u x_v] - [x_{uv}x_u x_v]^2}{(EG - F^2)^2}$$

Hence

Using the result of the above problem, prove that

$$K(EG - F^2)^2 = (F_{uv} - \frac{1}{2}E_{vv} - \frac{1}{2}G_{uu})(EG - F^2) + \det \begin{pmatrix} 0 & F_v - \frac{1}{2}G_u & \frac{1}{2}G_v \\ \frac{1}{2}E_u & E & F \\ F_u - \frac{1}{2}E_v & F & G \end{pmatrix} - \det \begin{pmatrix} 0 & \frac{1}{2}E_v & \frac{1}{2}G_u \\ \frac{1}{2}E_v & E & F \\ \frac{1}{2}G_u & F & G \end{pmatrix}$$

Note that this is a direct proof of Gauss' Theorem.

We note that

$$[abc][def] = \det \begin{pmatrix} d_1 & d_2 & d_3 \\ e_1 & e_2 & e_3 \\ f_1 & f_2 & f_3 \end{pmatrix} \det \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} = \det \begin{pmatrix} d_1 & d_2 & d_3 \\ e_1 & e_2 & e_3 \\ f_1 & f_2 & f_3 \end{pmatrix} \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = \det \begin{pmatrix} a \cdot d & b \cdot d & c \cdot d \\ a \cdot e & b \cdot e & c \cdot e \\ a \cdot f & b \cdot f & c \cdot f \end{pmatrix}$$

Thus from Problem 10.4 and the computations from Problem 10.3, we have

$$K(EG - F^2)^2 = \det \begin{pmatrix} x_{uu} \cdot x_{vv} & x_u \cdot x_{vv} & x_v \cdot x_{vv} \\ x_{uu} \cdot x_u & x_u \cdot x_u & x_v \cdot x_u \\ x_{uu} \cdot x_v & x_u \cdot x_v & x_v \cdot x_v \end{pmatrix} - \det \begin{pmatrix} x_{uv} \cdot x_{uv} & x_u \cdot x_{uv} & x_v \cdot x_{uv} \\ x_{uv} \cdot x_u & x_u \cdot x_u & x_v \cdot x_u \\ x_{uv} \cdot x_v & x_u \cdot x_v & x_v \cdot x_v \end{pmatrix} = \det \begin{pmatrix} x_{uu} \cdot x_{vv} & F_v - \frac{1}{2}G_u & \frac{1}{2}G_v \\ \frac{1}{2}E_u & E & F \\ F_u - \frac{1}{2}E_v & F & G \end{pmatrix} - \det \begin{pmatrix} x_{uv} \cdot x_{uv} & \frac{1}{2}E_v & \frac{1}{2}G_u \\ \frac{1}{2}E_v & E & F \\ \frac{1}{2}G_u & F & G \end{pmatrix}$$

it follows that

$$K(EG - F^2)^2 = (x_{uu} \cdot x_{vv} - x_{uv} \cdot x_{uv})(EG - F^2) - \det \begin{pmatrix} 0 & \frac{1}{2}E_v & \frac{1}{2}G_u \\ \frac{1}{2}E_v & E & F \\ \frac{1}{2}G_u & F & G \end{pmatrix} + \det \begin{pmatrix} \frac{1}{2}E_u & \frac{1}{2}G_u \\ F_u - \frac{1}{2}E_v & F \end{pmatrix}$$

Since both determinants have the common minor $\det \begin{pmatrix} E & F \\ F & G \end{pmatrix}$, it follows that

In Problem 10.3 we obtained $\mathbf{x}_{uu} \cdot \mathbf{x}_v = F_u - \frac{1}{2}E_v$ and $\mathbf{x}_{uv} \cdot \mathbf{x}_v = \frac{1}{2}G_u$. Hence

$$(F_u - \frac{1}{2}E_v)_v = (\mathbf{x}_{uu} \cdot \mathbf{x}_v)_v = \mathbf{x}_{uuv} \cdot \mathbf{x}_v + \mathbf{x}_{uv} \cdot \mathbf{x}_{vv}, \quad (\frac{1}{2}G_u)_u = (\mathbf{x}_{uv} \cdot \mathbf{x}_v)_u = \mathbf{x}_{uvu} \cdot \mathbf{x}_v + \mathbf{x}_{uv} \cdot \mathbf{x}_{uv}$$

Subtracting,

$$\mathbf{x}_{uu} \cdot \mathbf{x}_{vv} - \mathbf{x}_{uv} \cdot \mathbf{x}_{uv} = (F_u - \frac{1}{2}E_v)_v - \frac{1}{2}G_{uu} = F_{uv} - \frac{1}{2}E_{vv} - \frac{1}{2}G_{uu}$$

which gives the required result.

- 10.6. If the parameter curves on a patch are lines of curvature, show that the Codazzi-Mainardi equations (10.7) take the form

$$\frac{\partial \kappa_1}{\partial v} = \frac{1}{2} \frac{E_v}{E} (\kappa_2 - \kappa_1), \quad \frac{\partial \kappa_2}{\partial u} = \frac{1}{2} \frac{G_u}{G} (\kappa_1 - \kappa_2)$$

where κ_1 and κ_2 are the principal curvatures.

When the parameter curves are lines of curvature, $F = M = 0$. Thus equations (10.7) reduce to

$$L_v = L\Gamma_{12}^1 - N\Gamma_{11}^2 = \frac{LGE_v}{2EG} + \frac{NEE_v}{2EG} = \frac{1}{2}E_v \left(\frac{L}{E} + \frac{N}{G} \right)$$

$$\text{and} \quad N_u = -L\Gamma_{22}^1 + N\Gamma_{12}^2 = \frac{LGG_u}{2EG} + \frac{NEG_u}{2EG} = \frac{1}{2}G_u \left(\frac{L}{E} + \frac{N}{G} \right)$$

$$\text{or} \quad \left(\frac{L}{E} \right)_v = \frac{E_v}{2E} \left(\frac{N}{G} - \frac{L}{E} \right) \quad \text{and} \quad \left(\frac{N}{G} \right)_u = \frac{G_u}{2G} \left(\frac{L}{E} - \frac{N}{G} \right)$$

But from Theorem 9.13, page 186, if the parameter curves are lines of curvature, $\kappa_1 = L/E$ and $\kappa_2 = N/G$, which gives the required result.

- 10.7. Prove that there does not exist a compact surface in E^3 of class ≥ 2 with Gaussian curvature $K \leq 0$.

Suppose otherwise, i.e. suppose S is a compact surface of class ≥ 2 with $K \leq 0$ at each point. Now consider the real-valued function $f(P) = |\mathbf{x}|^2 = \mathbf{x} \cdot \mathbf{x}$, where $\mathbf{x}$ is the point P . We leave as an exercise for the reader to show that f is continuous on S . Hence from Theorem 6.9, page 110, f takes on its maximum, say, $f(P_0) = |\mathbf{x}_0|^2 = r^2$ at some point P_0 on S . Note that $r^2 > 0$. Otherwise, we would have $f \equiv 0$ on S , since $f \geq 0$ and r^2 is its maximum. But then S would consist of a single point $\mathbf{x} = 0$, which is impossible. Now let $\mathbf{x} = \mathbf{x}(u, v)$ be a patch on S containing P_0 such that the u and v parameter curves have principal directions at P_0 . Since $f(P) = f(\mathbf{x}(u, v))$ has a maximum at P_0 ,

$$\text{at } P_0. \quad \text{Also} \quad \frac{\partial f}{\partial u} = 2\mathbf{x} \cdot \mathbf{x}_u = 0 \quad \text{and} \quad \frac{\partial f}{\partial v} = 2\mathbf{x} \cdot \mathbf{x}_v = 0$$

$\frac{\partial^2 f}{\partial u^2} = 2\mathbf{x}_u \cdot \mathbf{x}_u + 2\mathbf{x} \cdot \mathbf{x}_{uu} \leq 0$ and $\frac{\partial^2 f}{\partial v^2} = 2\mathbf{x}_v \cdot \mathbf{x}_v + 2\mathbf{x} \cdot \mathbf{x}_{vv} \leq 0$ at P_0 . From the first two equations above we obtain that $\mathbf{x}$ is orthogonal to $\mathbf{x}_u$ and $\mathbf{x}_v$ at P_0 . Hence $\mathbf{N} = \pm \mathbf{x}/|\mathbf{x}| = \pm \mathbf{x}/r$ at P_0 . We may assume that the sense of $\mathbf{N}$ is such that $\mathbf{N} = \mathbf{x}/r$. Substituting into the second two equations above, we obtain that $\mathbf{x}_u \cdot \mathbf{x}_u + r\mathbf{N} \cdot \mathbf{x}_{uu} \leq 0$ and $\mathbf{x}_v \cdot \mathbf{x}_v + r\mathbf{N} \cdot \mathbf{x}_{vv} \leq 0$, or $E + rL \leq 0$ and $G + rN \leq 0$, or $L/E \leq -1/r < 0$ and $N/G \leq -1/r < 0$ at P_0 . Since the u and v parameter curves have principal directions at P_0 , it follows from Theorem 9.13, page 186, that $\kappa_1 = L/E$ and $\kappa_2 = N/G$, and hence from the above $K = \kappa_1 \kappa_2 = LN/EG \geq 1/r^2 > 0$ at P_0 which is again impossible since $K \leq 0$ on S . Thus the proposition is proved.

- 10.8. Prove that $f(P) = [\kappa_1(P) - \kappa_2(P)]^2$ is a continuous function on a surface.

Recall that the principal curvatures at a point P on a surface S depends on the orientation of the patch containing P , changing sign when there is a change in the sense of $\mathbf{N}$. Thus unless S is orientable, it may not be possible to define $\kappa_1(P)$ and $\kappa_2(P)$ themselves as continuous functions throughout S . Note however that f is independent of a change in sign of both κ_1 and κ_2 and hence is an intrinsic property of S , independent of the patch containing P .

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ENSOR ANALYSIS

To prove that f is continuous at a point P_0 . Since κ_1 and κ_2 are continuous functions of a point P_0 , we suppose that $\mathbf{x} = \mathbf{x}(u, v)$ is a patch containing P_0 . Since $f(P) = f(\mathbf{x}(u, v))$ is a continuous function of the first and second fundamental coefficients, that $|f(\mathbf{x}(u, v)) - f(\mathbf{x}(u_0, v_0))| < \epsilon$ for (u, v) in $S_{\delta_1}(u_0, v_0)$. From Problem 8.13, page 165, the image M of $S_{\delta_1}(u_0, v_0)$ on S is the intersection of an open set O in E^3 with S . It follows that there exists an $S_\delta(\mathbf{x}_0)$ in E^3 such that $S_\delta(\mathbf{x}_0) \cap S$ is contained in M . Thus for $\mathbf{x}$ in $S_\delta(\mathbf{x}_0) \cap S$ we have $|f(\mathbf{x}) - f(\mathbf{x}_0)| < \epsilon$. This shows that f is continuous at P_0 , which proves the proposition.

10.9.

Prove Hilbert's lemma: If at a point P_0 on a surface of sufficiently high class,
 (i) $\kappa_1(P_0)$ is a local maximum, (ii) $\kappa_2(P_0)$ local minimum (iii) $\kappa(P) > \kappa(P_0)$ then
 $K(P_0) \leq 0$.

Since $\kappa_1(P_0) \neq \kappa_2(P_0)$, P_0 is not an umbilical point. Thus from Theorem 9.10, page 185, there exists a patch $\mathbf{x} = \mathbf{x}(u, v)$ containing P_0 for which the parameter curves are lines of curvatures. From Problem 10.6 it follows that

$$\frac{\partial \kappa_1}{\partial v} = \frac{1}{2} \frac{E_v}{E} (\kappa_2 - \kappa_1) \quad \text{and} \quad \frac{\partial \kappa_2}{\partial v} = \frac{1}{2} \frac{G_v}{G} (\kappa_1 - \kappa_2)$$

$$= \frac{\partial^2 \kappa_1}{\partial v^2} \left(\frac{EE_{vv} - E_v^2}{E_v^2} \right) (\kappa_2 - \kappa_1) + \frac{1}{2} \frac{E_v}{E} (\kappa_2 - \kappa_1) v$$

$$= \frac{1}{2} \left(\frac{GG_{uu} - G_u^2}{G^2} \right) (\kappa_1 - \kappa_2) + \frac{1}{2} \frac{G_u}{G} (\kappa_1 - \kappa_2)_u$$

Since κ_1 and κ_2 are extreme values at P_0 , $\partial \kappa_1 / \partial v = \partial \kappa_2 / \partial u = 0$ at P_0 . Also $\kappa_1 \neq \kappa_2$ at P_0 . Thus from the first two equations above, $E_v = G_u = 0$ at P_0 . Substituting into the second two equations above, it follows that

$$\frac{\partial^2 \kappa_1}{\partial u^2} = \frac{1}{2} \frac{E_{vv}}{E} (\kappa_2 - \kappa_1) \quad \text{and} \quad \frac{\partial \kappa_2}{\partial u^2} = \frac{1}{2} \frac{G_{uu}}{G} (\kappa_1 - \kappa_2)$$

Since κ_1 is a maximum at P_0 , $\partial^2 \kappa_1 / \partial v^2 \leq 0$ at P_0 . Also $\kappa_1 > \kappa_2$ at P_0 and $E > 0$. Hence from the first equation above, $E_{vv} = 0$ at P_0 . Since κ_2 is a minimum at P_0 , $\partial^2 \kappa_2 / \partial u^2 \geq 0$ at P_0 . Also $G > 0$. Hence $G_{uu} = 0$ at P_0 . Finally, since the parameter curves are lines of curvature, we have $F = M = 0$. Also at P_0 , $E_v = 0$ and $G_u = 0$. From Problem 10.5 it follows that at P_0

$$K = -\frac{1}{2} \frac{E_{vv} + G_{uu}}{EG}$$

Since $E_m \leq 0$ and $G_{uu} \leq 0$ it follows that $K \leq 0$, which is the required result.

10.7: The only connected and compact surfaces of sufficiently small genus are spheres.

[illegible]

and Problem 10.23. Similarly

Also $\varrho_{\alpha m} \Gamma_{ik}^{\beta} \Gamma_{\beta j}^{\alpha} = \Gamma_{ik}^{\beta} \varrho_{\alpha m} \Gamma_{\beta j}^{\alpha} = (\Gamma_{ijm})_k = (\Gamma_{ijm})_k = (\Gamma_{\alpha jm} + \Gamma_{mj\alpha}) \Gamma_{ik}^{\alpha}$
 above $R_{mi/k} = (\Gamma_{ikm})_j - \Gamma_{ik}^{\alpha} \Gamma_{\alpha jm} = \Gamma_{ik}^{\alpha} \Gamma_{mj\alpha} - (\Gamma_{ijm})_k + \Gamma_{ij}^{\alpha} \Gamma_{\alpha km} + \Gamma_{ij}^{\alpha} \Gamma_{mk\alpha} + \Gamma_{ik}^{\beta} \Gamma_{\beta jm} - \Gamma_{ij}^{\beta} \Gamma_{\beta km}$
 which gives the required result. Thus substituting in the

Supplementary Problems

THEORY OF SURFACES

- 10.30. Obtain the Christoffel symbols Γ_{ij}^k for the cylinder $\mathbf{x} = \mathbf{y}(u) + v\mathbf{g}$, $\mathbf{g} = \text{constant}$, $|\mathbf{g}| = 1$.
 Ans. $\Gamma_{11}^1 = \mathbf{y}' \cdot \mathbf{y}''/|\mathbf{y}' \times \mathbf{g}|^2$, $\Gamma_{11}^2 = (\mathbf{g} \cdot \mathbf{y}')(\mathbf{y}' \cdot \mathbf{y}'')/|\mathbf{y}' \times \mathbf{g}|^2$, otherwise $\Gamma_{ij}^k = 0$.
- 10.31. Verify that the functions $E = 1 + 4u^2$, $F = -4uv$, $G = 1 + 4v^2$, $L = 2(4u^2 + 4v^2 + 1)^{-1/2}$, $M = 0$, $N = -2(4u^2 + 4v^2 + 1)^{-1/2}$ satisfy the compatibility conditions, equations (10.7) and (10.8), page 203.
- 10.32. Using the Weingarten equations, prove that $\mathbf{N}_u \times \mathbf{N}_v = (EG - F^2)^{1/2} \mathbf{KN}$.
- 10.33. Solve the Gauss-Weingarten equations for the surface whose fundamental coefficients are $E = 1$, $F = 0$, $G = 1$, $L = -1$, $M = 0$, $N = 0$. Ans. Circular cylinder of radius 1.
- 10.34. Derive Rodrigues' formula from the Weingarten equations.
- 10.35. If the parameter curves on a patch are orthogonal, prove that

$$K = -\frac{1}{\sqrt{EG}} \left[\frac{\partial}{\partial u} \left(\frac{1}{\sqrt{E}} \frac{\partial \sqrt{G}}{\partial u} \right) + \frac{\partial}{\partial v} \left(\frac{1}{\sqrt{G}} \frac{\partial \sqrt{E}}{\partial v} \right) \right]$$

- 10.36. Prove that $f(P) = \mathbf{x} \cdot \mathbf{x}$ is a continuous function of a point P on a surface.
- 10.37. Prove that the principal curvatures $\kappa_1(P)$ and $\kappa_2(P)$ are continuous functions of a point P on an oriented surface.
- 10.38. Prove Theorem 10.6: The only connected and closed surfaces of class ≥ 2 of which all points are planar points are planes.
- 10.39. Prove that the spheres are the only connected compact surfaces with positive Gaussian curvature and constant mean curvature.

TENSORS

- 10.40. If A_i and B_j are the components of two covariant vectors, show that the outer product $C_{ij} = A_i B_j$ are the components of a covariant tensor of rank 2.
- 10.41. Show that $\det \begin{pmatrix} a_1^1 & a_2^1 & a_3^1 \\ a_1^2 & a_2^2 & a_3^2 \\ a_1^3 & a_2^3 & a_3^3 \end{pmatrix} = e^{ijk} a_i^1 a_j^2 a_k^3$ where e^{ijk} is defined in Example 10.6, page 210.
- 10.42. Show that $\delta_{ij}^{\alpha\beta} A_{\alpha\beta} = A_{ij} - A_{ji}$.

- 10.43. If A^{ij} are the components of an absolute contravariant tensor and $A_{i\alpha}A^{\alpha j} = \delta_i^j$, show that $A^{i\alpha}$ are the components of an absolute covariant tensor. The two tensors are said to be *reciprocal*.
- 10.44. If A^{ij} and A_{ij} are the components of reciprocal symmetric tensors, and if x_i are the components of a covariant vector, show that $A_{ij}x^ix^j = A^{ij}x_ix_j$ where $x^i = A^{i\alpha}x_\alpha$.
- 10.45. Show that the quantities $e_{ijk} = e^{ijk}$, where the e^{ijk} are defined in Example 10.6(f), page 211, are the components of a covariant tensor of rank 3 and weight -1.
- 10.46. Let $\epsilon_{11} = 0$, $\epsilon_{12} = \sqrt{g}$, $\epsilon_{21} = -\sqrt{g}$, $\epsilon_{22} = 0$, where $g = g_{11}g_{22} - (g_{12})^2$. Show that the ϵ_{ij} , $i, j = 1, 2$ are the components of a skew-symmetric covariant tensor such that $\bar{\epsilon}_{11} = 0$, $\bar{\epsilon}_{12} = \sqrt{\bar{g}}$, $\bar{\epsilon}_{21} = -\sqrt{\bar{g}}$, $\bar{\epsilon}_{22} = 0$.
- 10.47. Let $e^{ij} = \epsilon_{\alpha\beta} g^{i\alpha} g^{j\beta}$ where $\epsilon_{\alpha\beta}$ is defined in the preceding problem. Show that $\epsilon^{11} = 0$, $\epsilon^{12} = 1/\sqrt{g}$, $\epsilon^{21} = -1/\sqrt{g}$, $\epsilon^{22} = 0$.
- 10.48. Show that $b_i^\beta b_{\beta j} - b_j^\beta b_{\beta i} = 0$, $i, j = 1, 2$.
- 10.49. Prove that $\bar{\Gamma}_{ij}^k = \left[\Gamma_{\alpha\beta}^\gamma \frac{\partial u^\alpha}{\partial \bar{u}^i} \frac{\partial u^\beta}{\partial \bar{u}^j} + \frac{\partial^2 u^\gamma}{\partial \bar{u}^i \partial \bar{u}^j} \right] \frac{\partial \bar{u}^k}{\partial u^\gamma}$.
- 10.50. Prove that $\frac{\partial g^{ij}}{\partial u^k} = -g^{\alpha j} \Gamma_{\alpha k}^i - g^{\alpha i} \Gamma_{\alpha k}^j$.
- 10.51. Show that
- $$\begin{aligned} R_{112}^1 &= R_{221}^2 = -R_{121}^1 = -R_{212}^2 = F \frac{LN - M^2}{EG - F^2} \\ R_{212}^1 &= -R_{221}^1 = G \frac{LN - M^2}{EG - F^2} \\ R_{121}^2 &= -R_{121}^2 = E \frac{LN - M^2}{EG - F^2} \end{aligned}$$
- and otherwise $R_{ijk}^p = 0$.
- 10.52. Prove that $R_{ijk}^p = \frac{\partial \Gamma_{ik}^p}{\partial u^j} - \frac{\partial \Gamma_{jk}^p}{\partial u^i} + \Gamma_{ik}^\alpha \Gamma_{\alpha j}^p - \Gamma_{ij}^\alpha \Gamma_{\alpha k}^p$.
- 10.53. Prove that $\frac{1}{2} \frac{\partial \log g}{\partial u^1} = \Gamma_{11}^1 + \Gamma_{12}^2$, $\frac{1}{2} \frac{\partial \log g}{\partial u^2} = \Gamma_{12}^1 + \Gamma_{22}^2$.